Unofficial notes\(^1\)

(Seminar on Commutative Banach Algebras - Vladimir P. Fonf)

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\(^1\)Editor's note

These notes were written during academic year 2014-2015. They are totally independent of the Professor's will. No guarantee is given regarding the completeness or correctness of this paper. In particular, it is highly probable that many errors (likely conceptual!) will occur because of the unskillfulness of the curator. Use the information contained herein at your own risk.

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Chapter 1

Commutative Banach Algebras

1.1 Basic definitions

Definition 1 (Banach algebra). A complex Banach space $X$ is called a (commutative) Banach algebra if there exist a product

$$X \times X \rightarrow X,$$

$$(x, y) \mapsto xy$$

such that

1. (commutativity) for all $x, y \in X$,

   $$xy = yx;$$

2. (associativity) for all $x, y, z \in X$,

   $$(xy) z = x (yz);$$

3. (distributivity) for all $x, y, z \in X$,

   $$x (y + z) = xy + xz;$$

4. (continuity) for all $\{x_n\}_{n \in \mathbb{N}} \subset X$, $x, y \in X$, there exists the limit

   $$\lim_{n \rightarrow +\infty} x_n y = xy.$$

Problem 2. The last property states that in Banach algebras the product is continuous with respect to both components. Does this implies that the product is continuous? We will see later on that the answer is yes.
Problem 3. Do Banach algebras always have a unit\textsuperscript{1}? Sadly, the answer is no. Luckily enough, though, there is a standard way to add a unit element to any Banach algebra.

**Proposition 4.** Let $X$ be a Banach algebra. Then, the set $\tilde{X} := \{(x, \lambda) \in X \times \mathbb{C}\}$ is a Banach algebra with respect to the operations
\[
\forall (x, \lambda), (y, \mu) \in \tilde{X}, \quad (x, \lambda) + (y, \mu) := (x + y, \lambda + \mu),
\]
\[
\forall (x, \lambda) \in \tilde{X}, \forall \mu \in \mathbb{C}, \quad \mu (x, \lambda) := (\mu x, \mu \lambda),
\]
\[
\forall (x, \lambda), (y, \mu) \in \tilde{X}, \quad (x, \lambda) (y, \mu) := (xy + \mu x + \lambda y, \lambda \mu)
\]
and the norm
\[
\forall (x, \lambda) \in \tilde{X}, \quad \|(x, \lambda)\|_{\tilde{X}} = \|x\|_X + |\lambda|;
\]
$\tilde{X}$ has unit $(0, 1)$ and admits a subalgebra isometric to $X$. The isometric embedding of $X$ in $\tilde{X}$ is given by
\[
\forall x \in X, \quad x \mapsto (x, 0).
\]

**Proof.** Trivial.

Remark 5 (Convention). Because of the previous proposition from now on we will always assume that Banach algebras have units. We will always use the letter $e$ to denote the unit element of a Banach algebra.

**Proposition 6.** Let $X$ be Banach space and $A, B : X \to X$ two bounded linear operators. Then, indicating with $|||\cdot|||$ the operator norm on $BL(X; X)$,
\[
|||AB||| \leq |||A||| \cdot |||B|||
\]

**Proof.** For all $x \in X$, $\|x\| = 1$,
\[
\|(AB)(x)\| = \|A(B(x))\| \\
\leq |||A||| \cdot |||B||| \cdot \|x\| \\
= 1
\]

Theorem 7. Let $(X, \|\cdot\|_X)$ be a Banach algebra. Then there exists a norm $\|\cdot\|$ equivalent to $\|\cdot\|_X$ such that, for all $x, y \in X$,
\[
\|xy\| \leq \|x\| \|y\| \tag{1.1.1}
\]
and $\|e\| = 1$.

\textsuperscript{1}I.e. an element $e \in X$ such that, for all $x \in X$, $ex = x$. 

1.1 Basic definitions

Proof. For all \( x \in X \), the map

\[
V_x : X \to X,
\]

\[
y \mapsto xy.
\]

is linear and continuous (because of the distributivity and the componentwise continuity the product). Then

\[
\mathcal{V} := \{ V_x \mid x \in X \} \subset BL(X; X) := \{ T : X \to X \mid T \text{ is bounded and linear} \}.
\]

Clearly \( \mathcal{V} \) is a complex linear space (distributivity again). It is well known that \( BL(X; X) \) is a Banach space. We want to prove that \( \mathcal{V} \) is a closed subspace of \( BL(X; X) \). Take any \( V \in \mathcal{V} \). Then there exists \( \{x_n\}_{n \in \mathbb{N}} \subset X \) such that

\[
V_{x_n} \overset{n \to +\infty}{\to} V.
\]

This means that, for \( n \to +\infty \)

\[
\sup_{y \in X} \{ |x_n y - V(y)| \} \to 0,
\]

i.e. for all \( y \in X \), if \( n \to +\infty \)

\[
x_n y \to V(y) \in X.
\]

This implies (by the componentwise continuity of the product and the completeness of \( X \)) that there exists \( x \in X \) such that

\[
x_n \to x.
\]

By the uniqueness of the limit and the componentwise continuity of the product, \( V = V_x \), thus \( V \in \mathcal{V} \) and consequently \( \mathcal{V} \) is a Banach space. It’s easy to check that \( \mathcal{V} \) is a Banach algebra with respect to the pointwise product\(^2\). Consider the mapping

\[
T : \mathcal{V} \to X,
\]

\[
V_x \mapsto x.
\]

Clearly \( T \) is linear. Note that, for all \( x \in X \)

\[
\|V_x\|_{\mathcal{V}} = \sup_{\|y\|_X = 1} \{ \|xy\|_X \} \geq \left\| \frac{x}{\|e\|_X} \right\|_X = \frac{1}{\|e\|_X} \|x\|_X.
\]

This shows that for all \( x \in X \),

\[
\|T(V_x)\|_X \leq \|e\|_X \|V_x\|_{\mathcal{V}},
\]

hence \( T \) is continuous. By the open mapping theorem \( T \) is an homeomorphism, so \( \|T^{-1}(\cdot)\|_{\mathcal{V}} \) and \( \|\cdot\|_X \) are equivalent norms. Clearly the unit of \( \mathcal{V} \) is \( V_e \) and

\[
\|V_e\|_{\mathcal{V}} = \sup_{\|x\|_X = 1} \{ \|ex\|_X \} = 1.
\]

Finally, the inequality (1.1.1) follows by the previous proposition. \( \Box \)

\(^2\) For proving the componentwise continuity of the product use the fact, proven above, that \( V_{x_n} \to V_x \) implies \( x_n \to x \).
Remark 8 (Convention). Because of the previous result from now on we will always assume that Banach algebras satisfy inequality (1.1.1) and that the unit has norm 1.

Example 9. Let’s now see some examples of Banach algebras.

1. The Banach space \((C [0,1], \|\cdot\|_\infty)\) is a Banach algebra with unit \(e \equiv 1\) with respect to the product \((fg) (\cdot) = f (\cdot) g (\cdot)\). The relations \(\|e\| = 1\) and

\[
\forall f, g \in C [0,1], \quad \|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty
\]

also (clearly) hold.

2. Let \(D \subset \mathbb{C}\) be the complex unit disk. The normed space

\[
D := \left\{ f \in C (\overline{D}; \mathbb{C}) \mid f|_D \text{ is holomorphic} \right\}, \sup_{\bar{D}} \{\cdot\}
\]

is a Banach algebra with unit \(e \equiv 1\) with respect to the pointwise product. \(D\) is called disk algebra. Clearly \(D\) satisfies (1.1.1).

3. The normed space

\[
\left( C^k ([0,1]; \mathbb{C}), \sum_{n=0}^{k} \frac{\| (\cdot)^{(n)} \|_\infty}{n!} \right)
\]

is a Banach algebra with unit \(e \equiv 1\) with respect to the pointwise product. It satisfies (1.1.1).

Corollary 10 (Continuity of the product). Let \(A\) be a Banach algebra. Let \(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset A\) and \(x, y \in A\). If \(x_n \to x\) and \(y_n \to y\), then \(x_ny_n \to xy\).

Proof. For all \(n \in \mathbb{N}\)

\[
\|x_ny_n - xy\| = \|x_ny_n - x_ny + x_ny - xy\| \\
\leq \|x_ny_n - x_ny\| + \|x_ny - xy\| \\
= \|x_n (y_n - y)\| + \|(x_n - x) y\| \\
\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\|.
\]

\[\Box\]

Theorem 11. Let \(A\) be a Banach algebra and \(G := \{x \in A \mid \exists x^{-1} \in A\}\). Then

1. \(G\) is open in \(A\);

2. the mapping

\[
\psi : G \to G, \quad x \mapsto \psi (x) := x^{-1}
\]

is continuous.
Proof.

1. Let’s start by proving that for all \( x \in A, \|x\| < 1 \), there exists \((e - x)^{-1} \in G\). I.e. there is a neighborhood of \( e \) consisting of invertible elements. Let’s prove that the series
\[
y := \sum_{k=0}^{\infty} x^k
\]
converges. By induction it’s easy to prove that, for all \( k \in \mathbb{N} \)
\[
\|x^k\| \leq \|x\|^k.
\]
Then, for all \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that, for all \( n, m \in \mathbb{N}, n_0 \leq n < m, \)
\[
\left\| \sum_{k=n+1}^{m} x^k \right\| \leq \sum_{k=n+1}^{m} \|x^k\| \leq \sum_{k=n+1}^{m} \|x\|^k < \varepsilon.
\]
Being \( A \) a Banach space, the sum converges.

Claim. \( y = (e - x)^{-1} \) i.e. \( y(e - x) = e. \)

Indeed
\[
y(e - x) = (e - x) \lim_{n \to +\infty} \sum_{k=0}^{n} x^k
\]
[The product is continuous]
\[
= \lim_{n \to +\infty} (e - x) \sum_{k=0}^{n} x^k
\]
\[
= \lim_{n \to +\infty} \left( \sum_{k=0}^{n} x^k - \sum_{k=1}^{n+1} x^k \right)
\]
\[
= \lim_{n \to +\infty} (e - x^{n+1})
\]
\[
= e
\]
Let’s finally prove that \( G \) is open. Take \( x \in G \) and \( h \in A \) such that
\[
\|h\| < \frac{1}{2 \|x^{-1}\|}.
\]
Then
\[
(x + h) = x(e + x^{-1}h).
\]
The right hand side is invertible by hypothesis and the first part of the proof, so \( x + h \) is invertible.
2. Take \( x \in G \) and \( h \in A \) so that
\[
\| h \| \leq \frac{1}{2 \| x^{-1} \|}.
\]
Then
\[
\| (x + h)^{-1} - x^{-1} \| = \| x^{-1} (e + x^{-1}h)^{-1} - x^{-1} \| \\
= \| x^{-1} \left( (e + x^{-1}h)^{-1} - e \right) \| \\
\leq \| x^{-1} \| \left( e + x^{-1}h \right)^{-1} - e \\
= (\ast).
\]
Being \( \| y \| < 1/2 \), by the previous part of the proof,
\[
(e + y)^{-1} = \sum_{k=0}^{+\infty} (-1)^k y^k.
\]
So
\[
(\ast) \leq \| x^{-1} \| \left| \sum_{k=1}^{+\infty} (-1)^k y^k \right| \\
= \| x^{-1} \| \left| y \sum_{k=0}^{+\infty} (-1)^{k+1} y^k \right| \\
\leq \| x^{-1} \| \| y \| \left| \sum_{k=0}^{+\infty} (-1)^{k+1} y^k \right| \\
\leq \| x^{-1} \| \| y \| \sum_{k=0}^{+\infty} \| y^k \| \\
\leq \| x^{-1} \| ^{2} \| h \| \sum_{k=0}^{+\infty} \frac{1}{2^k} \\
eq 2 \| x^{-1} \| ^{2} \| h \| ,
\]
and \( \psi \) is continuous.

\( \square \)

1.2 Basic properties of spectra

Definition 12 (Spectrum and resolvent set). Let \( A \) be a Banach algebra and \( x \in A \). The set
\[
\sigma (x) := \{ \lambda \in \mathbb{C} \mid x - \lambda e \text{ is not invertible} \}
\]
is called *spectrum of* \( x \). The set 
\[
\Omega(x) := \{ \lambda \in \mathbb{C} \mid \exists (x - \lambda e)^{-1} \}
\]
is called *resolvent set of* \( x \). For all \( \lambda \in \Omega(x) \), we will also use the notation 
\[
R_\lambda(x) := (x - \lambda e)^{-1}.
\]

**Lemma 13.** Let \( A \) be a Banach algebra and \( x \in A \). If \( \lambda \in \mathbb{C}, \ |\lambda| > \|x\| \), then \( \lambda \in \Omega(x) \). In other words
\[
\Omega(x) \supset \mathbb{C} \setminus B(0, \|x\|).
\]

*Proof.* Let’s fix \( \lambda \in \mathbb{C}, \ |\lambda| > \|x\| \). We need to prove the existence of \( R_\lambda(x) \). Being 
\[
x - \lambda e = \lambda \left( \frac{x}{\lambda} - e \right) = -\lambda \left( e - \frac{x}{\lambda} \right)
\]
and
\[
\left\| \frac{x}{\lambda} \right\| = \frac{\|x\|}{|\lambda|} < 1,
\]
by the proof of Theorem 11 on page 6 it follows that \( e - \frac{x}{\lambda} \) is invertible, hence there exists 
\[
(x - \lambda e)^{-1} = \lambda^{-1} \left( \frac{x}{\lambda} - e \right)^{-1}.
\]

**Corollary 14.** Let \( A \) be a Banach algebra and \( x \in A \). If \( \lambda \in \sigma(x) \), then \( |\lambda| \leq \|x\| \). In other words
\[
\sigma(x) \subset \overline{B(0, \|x\|)}.
\]

**Corollary 15.** Let \( A \) be a Banach algebra and \( x \in A \). Then the spectrum of \( x \) is a bounded set.

**Lemma 16.** Let \( A \) be a Banach algebra and \( x \in A \). Then the set \( \Omega(x) \) is open in \( \mathbb{C} \).

*Proof.* Suppose \( \lambda_0 \in \Omega(x) \). Let’s find a neighborhood of \( \lambda_0 \) in \( \Omega(x) \). By definition it exists \( (x - \lambda_0 e)^{-1} \). For all \( \varepsilon > 0 \), if \( \lambda \in \mathbb{C}, \ |\lambda - \lambda_0| < \varepsilon \), then 
\[
\|e - \lambda_0 x - (e - \lambda x)\| = |\lambda - \lambda_0| \|x\| < \varepsilon \|x\|.
\]
Being \( G \) open\(^3\), \( (x - \lambda e) \) is also invertible, hence \( \lambda \in \Omega(x) \).

**Corollary 17.** The set \( \sigma(x) \) is closed.

**Corollary 18.** The set \( \sigma(x) \) is compact.

\(^3 G \) is the set of invertible elements in \( A \).
1.2 Basic properties of spectra

1.2.1 Abstract holomorphic functions

**Definition 19** (Abstract holomorphic function). Let $A$ a Banach algebra, $D \subset \mathbb{C}$ open and $f : D \to A$. Given $z_0 \in D$, we say that $f$ is *abstractly holomorphic* in $z_0$ if the following limit exists:

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$ 

If $f$ is abstractly holomorphic in every point of $D$ we say that $f$ is an abstract holomorphic function. If $f$ is an abstract holomorphic function and $D = \mathbb{C}$ we say that $f$ is an abstract entire function.

**Remark 20.** Almost every result which holds for holomorphic functions also holds for abstract holomorphic functions.

**Theorem 21** (Dunford). Let $A$ a Banach algebra, $D \subset \mathbb{C}$ open and $f : D \to A$. Then $f$ is an abstract holomorphic function if and only if for all $\alpha \in A^*$, $\alpha \circ f : D \to \mathbb{C}$ is a (regular) holomorphic function.

**Theorem 22.** Let $A$ a Banach algebra, $D \subset \mathbb{C}$ open and $f : D \to A$. Then for all $z_0 \in D$ there exists $r > 0$ such that, for all $z \in B(z_0, r)$,

$$f(z) = \sum_{k=0}^{+\infty} f^{(k)}(z_0) (z - z_0)^k.$$ 

**Notation 23** (Radius of convergence). What is the radius of convergence of the previous power series? Without loss of generality, suppose $z_0 = 0$. We need to study the convergence of the series

$$z \mapsto \sum_{n=0}^{+\infty} c_n z^n.$$ 

Defining

$$R := \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{\|c_n\}},$$ 

by the general theory of power series, it is possible to state that the series converges in $B(0, R)$. But let’s look at this problem from a different prospective. Fix $c \in A \setminus \{0\}$ and consider the complex power series

$$z \mapsto \sum_{n=0}^{+\infty} c^n z^n.$$ 

Clearly, for all $z \in \mathbb{C}$ such that $\|cz\| < 1$, i.e., for all $z \in B(0, 1/\|c\|)$, the series converges. Since by the general theory if the series converges in $z \in \mathbb{C}$, then $z \in B(0, R)$, the following inequality should hold:

$$\frac{1}{\|c\|} \leq \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{\|c^n\}}} \quad (1.2.1)$$
This is an easy exercise, in fact for all $n \in \mathbb{N}$
\[
\sqrt[n]{c^n} \leq \sqrt[n]{\|c\|^n} = \|c\|.
\]
The following proposition proves a useful fact. The $\limsup$ in (1.2.1) always exists as a limit.

**Proposition 24.** Let $A$ be a Banach algebra and $x \in A$. Then there exists the limit
\[
\lim_{n \to +\infty} \sqrt[n]{\|x^n\|}.
\]

**Proof.** For all $n \in \mathbb{N}$, call $\alpha_n := \|x^n\|$. For all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$ there exist $m, \ell \in \mathbb{N}$ such that $n = mk + \ell$ and consequently
\[
\alpha_n = \|x^n\| = \|x^{km+\ell}\| = \| (x^k)^m x^\ell \| \leq \| (x^k)^m \| \|x^\ell\| \leq \|x^k\|^m \|x^\ell\| = \alpha_k^m \alpha_\ell,
\]

hence
\[
(\alpha_n)^\frac{1}{n} \leq \alpha_k^{m/n} \alpha_\ell^{1/n}.
\]

Note that if $n, k, m, \ell \in \mathbb{N}$ satisfy $n = mk + \ell$, then $\frac{m}{n} = \frac{1}{k} - \frac{\ell}{kn}$ and
\[
(\alpha_n)^\frac{1}{n} \leq \alpha_k^{-\ell/k} \alpha_\ell^{1/n}.
\]

Since $\ell$ is bounded by 0 and $k$ (it’s the rest in the Euclidean division), taking both sides to the $\limsup$, we derive, for all $k \in \mathbb{N}$,
\[
\limsup_{n \to +\infty} (\alpha_n)^\frac{1}{n} \leq \alpha_k^{1/k}.
\]

Now, taking both member to the $\liminf$, we get
\[
\limsup_{n \to +\infty} (\alpha_n)^\frac{1}{n} \leq \liminf_{k \to +\infty} \alpha_k^{1/k}
\]
which proves the existence of the limit. $\square$

**Notation 25** ($R_\lambda$). Let $A$ be a commutative Banach algebra. Whenever $x \in A$ is fixed, for all $\lambda \in \Omega(x)$ we will write $R_\lambda$ instead of $R_\lambda(x)$ in order to shorten the notation.

**Lemma 26** (Hilbert). Let $A$ be commutative Banach algebra and $x \in A$. Then for all $\lambda, \mu \in \Omega(x)$
\[
R_\lambda - R_\mu = (\lambda - \mu) R_\mu R_\lambda.
\]
1.2 Basic properties of spectra

Proof. For all \( \lambda, \mu \in \Omega(x) \),

\[
R_\lambda [(x - \mu e) - (x - \lambda e)] R_\mu = R_\lambda [(\lambda - \mu) e] R_\mu.
\]

\[\square\]

Corollary 27. Let \( A \) be commutative Banach algebra and \( x \in A \). Then the function \( \lambda \mapsto R_\lambda \) is holomorphic on the whole \( \Omega(x) \).

Proof. For all \( \lambda, \mu \in \Omega(x) \)

\[
\frac{R_\lambda - R_\mu}{\lambda - \mu} = R_\mu R_\lambda.
\]

Since the inverse function \((\cdot) \mapsto (\cdot)^{-1}\) is continuous where it’s defined, for all \( \mu \in \Omega(x) \) there exists the limit

\[
\lim_{\lambda \to \mu} R_\lambda = R_\mu.
\]

\[\square\]

Corollary 28. Let \( A \) be commutative Banach algebra and \( x \in A \). Then. We have

\[
\frac{\partial}{\partial \lambda} (\lambda \mapsto R_\lambda) = (\lambda \mapsto R_\lambda^2).
\]

Theorem 29 (Liouville). Let \( A \) be a commutative Banach algebra and \( f : \mathbb{C} \to A \) an abstract entire function. If \( f \) is is bounded, the \( f \) is constant.

Exercise 30. Prove the previous result.

Theorem 31. Let \( A \neq \{0\} \) be commutative Banach algebra. Then For all \( x \in A, \sigma(x) \neq \emptyset \).

Proof. Assume the contrary. Then there is \( x \in X \) s.t. \( \sigma_x = \emptyset \), i.e. so \( \Omega(x) = \mathbb{C} \). Hence \( \lambda \mapsto R_\lambda = (x - \lambda e)^{-1} \) is an entire function. Fix \( R > 0 \).

- For all \( \lambda \in \mathbb{C} \setminus \overline{B(0, R)} \),

\[
R_\lambda = \frac{1}{\lambda} \left( \frac{x}{\lambda} - e \right)^{-1}.
\]

Note that if \( |\lambda| \to +\infty \),

\[
\frac{x}{\lambda} - e \to e.
\]

So, if \( |\lambda| \to +\infty \),

\[
\left( \frac{x}{\lambda} - e \right)^{-1} \to (-e)^{-1} = -e.
\]

This implies that the function \( \lambda \mapsto (\frac{x}{\lambda} - e)^{-1} \) is bounded. For \( |\lambda| \to +\infty \) it follows

\[
\|R_\lambda\| = \frac{1}{|\lambda|} \left\| \left( \frac{x}{\lambda} - e \right)^{-1} \right\| \to 0 \text{ bounded},
\]

hence \( \lambda \mapsto \|R_\lambda\| \) is bounded on \( \mathbb{C} \setminus \overline{B(0, R)} \).
The function $\lambda \mapsto \|R_\lambda\|$ is continuous (composition of norm-continuous - with $\lambda \mapsto R_\lambda$ - holomorphic) on the compact set $B(0, R)$, so it is bounded on $B(0, R)$ also.

Putting together these two points, it follows that $\lambda \mapsto \|R_\lambda\|$ is bounded on $\mathbb{C}$.

By Liouville’s Theorem, it’s constant. Since, for $|\lambda| \to +\infty$, $R_\lambda \to 0$, necessarily $\lambda \mapsto R_\lambda \equiv 0$. This is a contradiction because for all $\lambda \in \mathbb{C}$ $R_\lambda$ is an inverse, and it can’t happen that for some $\lambda \in \mathbb{C}$

$$e = (x - \lambda e) (x - \lambda e)^{-1} = 0.$$

\[ \square \]

**Corollary 32** (Gelfand-Mazur Theorem). Let $A \neq \{0\}$ be a Banach algebra. If each $x \in A \setminus \{0\}$ is invertible, then $A$ is isometric to $\mathbb{C}$.

**Proof.** For all $x \in A$, by hypothesis and the definition of spectrum, $\sigma(x)$ is a singleton containing the only $\lambda \in \mathbb{C}$ satisfying

$$x - \lambda e = 0.$$

So for all $x \in A \setminus \{0\}$ there exist a unique $\lambda \in \mathbb{C}$ such that $x = \lambda e$. The mapping

$$A \to \mathbb{C},
  x \mapsto \lambda$$

is an isometry. \[ \square \]

### 1.3 Ideals

**Definition 33** (Ideal). Let $A$ be a Banach algebra and $I \subset A$. $I$ is called (non-trivial) ideal if

1. $I$ is a subspace\(^4\) of $A$;
2. for all $x \in A$ and for all $y \in I$, $xy \in I$;
3. $I \neq \{0\}$ and $I \neq A$.

**Remark 34.** The last axiom is not necessary for the definition of ideal. We still prefer to add it to the definition of ideal to avoid trivial cases and make the development of the theory cleaner.

**Example 35.** In $C[0,1]$ is easy to check that the following set is an ideal

$I := \left\{ f \in C[0,1] \mid f_{[0,1]} \equiv 0 \right\}$.

\(^4\)Not necessarily closed.
1.3 Ideals

**Lemma 36.** Let $A$ be a Banach algebra. If $I \subset A$ is an ideal, then for all $x \in I$, $x$ is not invertible.

*Proof.* Trivial. \( \square \)

**Lemma 37.** Let $A$ be a Banach algebra. For all non-invertible $x \in A \setminus \{0\}$ there is an ideal $I \subset A$ such that $x \in I$.

*Proof.* For all $x \in A$ not invertible, consider $I = \{ xy \mid y \in A \}$.

*Corollary 38.* Let $A$ be a Banach algebra. The following statements are equivalent:

1. all $x \in A \setminus \{0\}$ are invertible
2. no ideals exist in $A$.

*Proof.* Follows directly from the previous two lemmas. \( \square \)

**Definition 39** (Maximal). Let $A$ be a Banach algebra. An ideal $M \subset A$ is called *maximal* if it is maximal with respect to the inclusion.

**Example 40.** Let $A = C[0, 1]$. For all $t_0 \in [0, 1]$, define the ideal

$$ M_{t_0} := \{ f \in C[0, 1] \mid f(t_0) = 0 \}. $$

*Claim.* For all $t_0 \in [0, 1]$, $M_{t_0}$ is maximal.

*Proof.* Fix $t_0 \in [0, 1]$. Note that,

$$ C[0, 1] = C + M_{t_0}, $$

in fact, for all $f \in C[0, 1]$,

$$ f = f(t_0) + f - f(t_0) \in M_{t_0}. $$

This implies that the ideal is maximal. \( \square \)

*Claim.* For each maximal ideal $M \subset C[0, 1]$ there exists a unique $t_0 \in [0, 1]$ such that $M = M_{t_0}$.

*Proof.* Fix a maximal ideal $M \subset C[0, 1]$. Assume to the contrary that for all $\tau \in [0, 1]$ there is $f_\tau \in M$ such that $f_\tau(\tau) \neq 0$. This implies that for all $\tau \in [0, 1]$ there is $f_\tau \in M$, a neighborhood $U_\tau$ of $\tau$ and a positive constant $\delta_\tau > 0$ such that, for all $t \in U_\tau$,

$$ |f_\tau(t)| > \delta_\tau > 0. $$

Being $\{ U_\tau \}_{\tau \in [0, 1]}$ an open cover of the compact set $[0, 1]$, there exists $\tau_{k_1}, \ldots, \tau_{k_n} \in [0, 1]$ such that

$$ [0, 1] \subset \bigcup_{k=1}^{n} U_{\tau_k}. $$
1.3 Ideals

Note that if \( f \in \mathcal{C}[0, 1] \), also the complex conjugate \( \overline{f} \in \mathcal{C}[0, 1] \). Being \( M \) an ideal, this means that if \( f \in M \), also the real function \( f^2 = |f|^2 \in M \). Hence

\[
M \cap \mathbb{R} \ni \sum_{k=1}^{n} f_{\tau_k} \overline{f}_{\tau_k} = \sum_{k=1}^{n} \left| f_{\tau_k} \right|^2 \geq \min_{k \in \{1, \ldots, n\}} \{ |\delta_{\tau_k}| \} > 0.
\]

This is a contradiction because, being strictly positive, \( f \in M \) is invertible. It follows that there exists \( t_0 \in [0, 1] \) such that all the functions in \( M \) vanish in \( t_0 \), i.e. \( M \subset M_{t_0} \). Being \( M \) maximal, \( M = M_{t_0} \). Noting that \( t_0, t_1 \in [0, 1], t_0 \neq t_1 \) implies \( M_{t_0} \neq M_{t_1} \), the uniqueness is trivial. \( \square \)

**Proposition 41.** Let \( A \) be a commutative Banach algebra. If \( I \subset A \) is an ideal, the closure \( I^A \) is also an ideal.

**Proof.** First of all, let’s prove that \( I^A \) is a linear subspace of \( A \). For all \( x, y \in I^A \) there exist \( \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset I \) such that, for \( n \to +\infty \),

\[
x_n \to x, \quad y_n \to y.
\]

By the continuity of the sum,

\[
x + y = \lim_{n \to +\infty} (x_n + y_n),
\]

hence \( x + y \in I^A \). Now take \( x \in I^A \) and \( y \in X \). There exists \( \{x_n\}_{n \in \mathbb{N}} \subset I \) such that \( x_n \to x \). Because of the continuity of the product, if \( n \to +\infty \), \( x_n y \to xy \).

Since \( I \) is an ideal, for all \( n \in \mathbb{N}, x_n y \in I \), hence \( xy \in I^A \). Finally, let’s prove that \( I^A \) is non trivial. Call \( G := \{ x \in A | x \text{ is invertible} \} \). Being an ideal, \( I \subset A \setminus G \), which is a closed set\(^6\), so

\[
\emptyset \subset I \subset I^A \subset A \setminus G \subset A.
\]

\( \square \)

**Corollary 42.** Let \( A \) be a commutative Banach algebra. If \( M \subset A \) is a maximal ideal, \( M \) is closed.

**Theorem 43.** Let \( A \) be a commutative Banach algebra and \( I \subset A \) an ideal. Then there exists a maximal ideal \( M \subset A \) containing \( I \).

**Proof.** Assume that \( I \) is not maximal. Consider the family \( \mathcal{M} \) of all ideals containing \( I \), partially ordered by inclusion. Clearly \( \mathcal{M} \) is not empty (\( I \in \mathcal{M} \)). Note that if any linearly ordered\(^7\) subfamily \( \{I_{\lambda}\}_{\lambda \in \Lambda} \subset \mathcal{M} \) has an upper bound, we are done (Zorn’s Lemma). Denote \( I_0 = \bigcup_{\alpha} I_{\alpha} \). By definition of \( \mathcal{M} \), \( I_0 \supset I \) and clearly \( I_0 \) is an ideal upper bound to \( \{I_{\lambda}\}_{\lambda \in \Lambda} \).

\(^5\)Theorem 11 states that \( G \) is open.

\(^6\)\( \{I_{\lambda}\}_{\lambda \in \Lambda} \subset \mathcal{M} \) is linearly ordered if and only if, for all \( \lambda_1, \lambda_2 \in \Lambda, I_{\lambda_1} \subset I_{\lambda_2} \) or \( I_{\lambda_2} \subset I_{\lambda_1} \).
Proposition 44. Let $A$ be a commutative Banach algebra, $G := \{ x \in A \mid x \text{ is invertible} \}$ and $x \in A$. The following statements are equivalent:

1. $x \in G$;
2. for all $M \subset A$ maximal ideals, $x \notin M$.

Proof. Both implications are direct consequences of previously proven results. \qed

Example 45. In Example 40 on page 14 we showed all maximal ideals in $C[0,1]$.

1.3.1 Multiplicative functionals

Definition 46 (Multiplicative functional). Let $A$ be a Banach algebra. A linear functional $f \in A' \setminus \{0\}$ is called multiplicative if, for all $x, y \in A$,

$$f(xy) = f(x)f(y).$$

Example 47. Let $A = C[0,1]$ and $t_0 \in [0,1]$. The evaluation functional

$$V_{t_0} : C[0,1] \to \mathbb{C},$$

$$f \mapsto V_{t_0}(f) := f(t_0)$$

is multiplicative. Indeed for all $f, g \in A$

$$V_{t_0}(fg) = (fg)(t_0) = f(t_0)g(t_0) = V_{t_0}(f)V_{t_0}(g).$$

Proposition 48. Let $A$ be a Banach algebra and $f \in A'$ a multiplicative functional. Then

1. $f(e) = 1$;
2. for all $x \in G$, $f(x) \neq 0$ and

$$f(x^{-1}) = \frac{1}{f(x)}.$$

Proof.

1. Trivial: $f(e) = f(e \cdot e)$.
2. Trivial: for all $x \in G$, $f(e) = f(x \cdot x^{-1})$. \qed

Proposition 49. Let $A$ be a Banach algebra and $f \in A'$ a multiplicative functional. If $x \in A$, $\|x\| < 1$, then $|f(x)| < 1$.

\footnote{Not necessarily bounded.}
1.3 Ideals

Proof. Call $\lambda := f(x)$. Assume to the contrary that $|\lambda| \geq 1$ and consider

$$x - \lambda e = \lambda \left(\frac{x}{\lambda} - e\right).$$

Since

$$\|\frac{x}{\lambda}\| < 1,$$

the right hand side is invertible and so is the left hand side. By the previous Proposition $f(x - \lambda e) \neq 0$ which leads to

$$f(x) \neq \lambda.$$

Absurd.

Remark 50. The previous result says that multiplicative functionals map the unit ball of the Banach algebra into the unit complex ball.

Corollary 51. Let $A$ be a Banach algebra and $f \in A'$ a multiplicative functional. Then

$$\|f\| = 1.$$

Proof. Because of the previous proposition

$$\|f\| = \sup_{x \in B_A} |f(x)| \leq 1.$$

Proposition 48 on the preceding page clinches the result.

Remark 52. The previous result says that multiplicative functionals are always continuous!

Proposition 53. Let $A$ be a Banach algebra and $f \in A^*$ a multiplicative functional. Then $\ker(f)$ is a maximal ideal.

Proof. It is well known that kernels of continuous linear functionals are closed subspace of codim = 1. It’s easy to check that $\ker(f)$ is an ideal. Having codimension 1, $\ker(f)$ is clearly maximal.$^8$

1.3.2 Quotient algebra

Proposition 54. Let $A$ be a Banach algebra and $I \subset A$ a closed ideal. Then the set

$$A/I := \{x + I | x \in A\}$$

is a Banach algebra with respect to the operations

$$\forall (x + I), (y + I) \in A/I, (x + I) + (y + I) := (x + y) + I,$$

$$\forall (x + I) \in A/I, \forall \mu \in \mathbb{C}, \mu (x + I) := \mu x + I,$$

$$\forall (x + I) (y + I) \in A/I (x + I) (y + I) := xy + I.$$

$^8$It can’t be enlarged without getting the whole $A$. 

and the norm
\[ \forall (x + I) \in A/I, \|x + I\| = \inf_{u \in x + I} \{\|u\|_A\} . \]

Furthermore, calling \( e \) the unit of \( A \),

1. \( A/I \) has unit \( e + I \);
2. \( \|e + I\| = 1 \);
3. for all \((x + I), (y + I) \in A/I\),
\[ \|(x + I)(y + I)\| \leq \|x + I\|\|y + I\| . \]

**Proof.** Easy and boring. \( \square \)

**Definition 55** (Quotient algebra). Let \( A \) be a Banach algebra and \( I \subset A \) a closed ideal. The Banach algebra \( A/I \) defined in the previous proposition is called *quotient algebra*.

**Remark 56.** Proposition 54 on the previous page states that quotients of a closed ideals are Banach algebras.

**Problem 57.** What can be said of the quotient of a Banach algebra with a maximal ideal?

**Exercise 58.** Let \( A \) be a Banach algebra and \( I \subset A \) a closed ideal. Prove that the quotient map \( q: A \to A/I \) is multiplicative, i.e. for all \( x, y \in A \),
\[ q(xy) = q(x)q(y) . \]

**Exercise 59.** Let \( A \) be a Banach algebra, \( I \subset A \) a closed ideal and \( q: A \to A/I \). Prove that for all ideal \( J \subset A/I \), \( q^{-1}(J) \subset A \) is an ideal.

**Lemma 60.** Let \( A \) be a Banach algebra, \( I \subset A \) a closed ideal, \( J' \subset A/I \) a closed ideal on the quotient algebra and \( q: A \to A/I \) the quotient map. Then \( J := q^{-1}(J') \) is a closed ideal containing \( I \).

**Proof.** Clearly \( J \supset q^{-1}(0) = I \). By the continuity of \( q \) and the previous exercise \( J \) is a closed ideal. \( \square \)

**Proposition 61.** Let \( A \) be a Banach algebra and \( M \subset A \) a maximal ideal. Then \( A/M \) has no ideals.

**Proof.** Let \( q: A \to A/M \) be the quotient map. Assume to the contrary that exists \( J \in A/M \) ideal. Without loss of generality, \( J \) is closed. By the previous lemma, \( q^{-1}(J) \) is an ideal containing \( M \). Absurd. \( \square \)
Corollary 62. Let $A$ be a Banach algebra and $M \subset A$ a maximal ideal. Then $A/M$ is a field.

Proof. It follows directly from Proposition 44 on page 16. \qed

Corollary 63. Let $A$ be a Banach algebra and $M \subset A$ a maximal ideal. Then $A/M$ is isometric to $\mathbb{C}$.

Proof. It follows directly from Gelfand-Mazur Theorem (Corollary 32 on page 13). \qed

Corollary 64. Let $A$ be a Banach algebra and $M \subset A$ a maximal ideal. Then $q : A \to A/M = \mathbb{C}$ is a multiplicative functional with $\ker(q) = M$.

Definition 65 ($\Delta$). Let $A$ be a Banach algebra. We define the set

$$\Delta := \{ \alpha : A \to \mathbb{C} \mid \alpha \text{ is multiplicative} \}.$$

Remark 66. Corollary 64 states that any maximal ideal is the kernel of some multiplicative functional. On the other hand, Proposition 53 on page 17 stated that kernels of multiplicative functionals were maximal ideals. This leads to the following, unexpected, result.

Corollary 67. Let $A$ be a Banach algebra and $M$ the set of maximal ideals in $A$. There is a one-to-one correspondence

$$\Phi : M \leftrightarrow \Delta.$$

Example 68. In $C[0,1]$, $\text{Card}(\Delta) = \text{Card}[0,1]$.

Definition 69 (Gelfand topology). Let $A$ be a Banach algebra. The topology induced on $\Delta$ by the $w^*$ topology of $A^*$ is called Gelfand topology.

Remark 70. From now on we will always consider $\Delta$ as a topological spaces endowed with the Gelfand topology. As the follow result shows, this choice makes $\Delta$ a compact (Hausdorff) topological space.

Theorem 71. Let $A$ be a Banach algebra. Then $\Delta$ is $w^*$-compact.

Proof. Because of Corollary 51 on page 17, $\Delta \subset B_{A^*}$, which is $w^*$-compact by Banach-Alaoglu theorem. Then we just need to prove that $\Delta$ is $w^*$-closed. We do that using nets. Fix a net $\{\alpha_{\lambda}\}_{\lambda \in \Lambda} \subset \Delta$ and a functional $\alpha \in A'$ such that

$$\alpha_{\lambda} \xrightarrow{w^*} \alpha.$$

Remember that the $w^*$-convergence of a net is just its pointwise convergence:

$$\alpha_{\lambda} \xrightarrow{w^*} \alpha \iff \forall x \in A, \ \alpha_{\lambda}(x) \to \alpha(x).$$

We want to prove that $\alpha \in \Delta$. Clearly $\alpha \neq 0$, indeed

$$1 = \alpha_{\lambda}(e) \to \alpha(e) \neq 0.$$
By the continuity of the product, it’s also easy to show that for all \( x, y \in A \), \( \alpha(xy) = \alpha(x) \alpha(y) \), indeed
\[
\frac{\alpha(x)}{\alpha(y)} = \frac{\alpha(x)}{\alpha(y)}
\]
\[
\alpha(x) = \alpha(x).
\]

**Definition 72** (Gelfand transform). Let \( A \) be a Banach algebra. The mapping
\[
\Lambda: A \rightarrow C(\Delta; \mathbb{C}),
\]
\[
x \mapsto \Lambda x := \hat{x},
\]
where
\[
\hat{x}: \Delta \rightarrow \mathbb{C},
\]
\[
\alpha \mapsto \hat{x} \alpha := \alpha(x)
\]
is called Gelfand transform.

**Proposition 73.** Let \( A \) be a Banach algebra. Then the Gelfand transform \( \Lambda \) is
a linear multiplicative mapping, i.e. for all \( x, y \in A \)
\[
\Lambda(x+y) = \hat{x} + \hat{y},
\]
\[
\Lambda(xy) = \hat{x} \hat{y}.
\]

*Proof.* Trivial. \( \square \)

**Proposition 74.** Let \( A \) be a Banach algebra. Then
\[
\|\|\Lambda\|\| = 1.
\]

*Proof.* We want to prove that
\[
\sup_{x \in B_A} \left\{ \|\hat{x}\|_{C(\Delta; \mathbb{C})} \right\} = 1.
\]
This follows from noting that for all \( x \in B_A \)
\[
\|\hat{x}\|_{C(\Delta; \mathbb{C})} = \sup_{\alpha \in \Delta} \{ |\alpha(x)| \} \leq \|x\|_A \leq 1
\]
and
\[
\|\hat{e}\|_{C(\Delta; \mathbb{C})} = 1.
\]

*Proof.* \( \square \)

**Proposition 75.** Let \( A \) be a Banach algebra. The Gelfand topology on \( \Delta \) is
the weakest topology that makes every \( \hat{x} \in \hat{A} \) continuous.
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Proof. If for all \( x \in A \),

\[
\gamma_x : A^* \to A, \\
\alpha \mapsto \gamma_x (\alpha) := \alpha (x),
\]
a base\(^9\) for the \( w^* \) topology of \( A^* \) is given by

\[
\left\{ \bigcap_{k=1}^n \{ \alpha \in A^* | \gamma_{x_k} (\alpha) > \varepsilon \} \right\}_{\varepsilon > 0}.
\]

This means that a base for the Gelfand topology of \( \Delta \) is simply

\[
\left\{ \bigcap_{k=1}^n \{ \alpha \in A^* | \gamma_{x_k} (\alpha) > \varepsilon \} \cap \Delta \right\}_{\varepsilon > 0} = \left\{ \bigcap_{k=1}^n \{ \alpha \in \Delta | \gamma_{x_k} (\alpha) > \varepsilon \} \right\}_{\varepsilon > 0} = \left\{ \bigcap_{k=1}^n \{ \alpha \in \Delta | \alpha (x_k) > \varepsilon \} \right\}_{\varepsilon > 0} = \left\{ \bigcap_{k=1}^n \{ \alpha \in \Delta | \hat{x}_k (\alpha) > \varepsilon \} \right\}_{\varepsilon > 0}.
\]

Theorem 76. Let \( A \) be a Banach algebra and \( x \in A \). Then\(^10\)

\[
\sigma (x) = \{ \alpha (x) \}_{\alpha \in \Delta}.
\]

Proof. Let’s start by proving that \( \sigma (x) \subset \{ \alpha (x) \}_{\alpha \in \Delta} \). Take \( \lambda \in \sigma (x) \). Then \( x - \lambda e \) is not invertible. So it lies in some maximal ideal, kernel of some multiplicative functional \( \alpha \). Since \( \alpha (x - \lambda e) = 0 \) it immediately follows that \( \alpha (x) = \lambda \). Vice versa, take an arbitrary \( \alpha \in \Delta \) and consider \( \lambda := \alpha (x) \in \mathbb{C} \). By definition \( \lambda \) satisfies

\[
0 = \alpha (x) - \lambda = \alpha (x - \lambda e),
\]

hence \( x - \lambda e \in \ker (\alpha) \), which is a maximal ideal. By Proposition 44 on page 16 it follows that \( \lambda \in \sigma (x) \).

Example 77. Consider \( A = \mathbb{C} [0,1] \). For all \( t_0 \in [0,1] \) consider the Dirac \( \delta \) distribution

\[
\delta_{t_0} : \mathbb{C} [0,1] \to \mathbb{C}, \\
f \mapsto f (t_0).
\]

\(^9\)For the open neighborhoods of the origin.
\(^10\)Pure mathemagics!
1.3 Ideals

Clearly, for all \( t_0 \in [0, 1] \), \( \delta_{t_0} \) is a multiplicative functional. With the same notation as Example 40 on page 14, for all \( t_0 \in [0, 1] \),

\[
M_{t_0} = \ker (\delta_{t_0}).
\]

Being maximal ideals in a one-to-one correspondence with multiplicative functionals, it follows

\[
\Delta = \{ \delta_{t_0} \}_{t_0 \in [0, 1]}.
\]

By Theorem 76 on the previous page, for all \( f \in C[0, 1] \),

\[
\sigma(f) = \{ \delta_{t_0}(f) \}_{t_0 \in [0, 1]} = \{ f(t_0) \}_{t_0 \in [0, 1]} = f([0, 1]).
\]

This makes perfect sense! For all \( f \in C[0, 1] \), \( \lambda \in \sigma(f) \) if and only if \( f - \lambda e \) is not invertible, which happens if and only if \( f - \lambda e \) is null at some point, i.e. if and only if there exists \( t_0 \in [0, 1] \) such that

\[
f(t_0) - \lambda e(t_0) = 0 \iff \lambda = f(t_0).
\]

Example 78. \( L^1[0, 1] \) is a Banach algebra with respect to the convolution product

\[
\forall f, g \in L^1[0, 1], x \mapsto (f * g)(x) = \int_0^x f(t)g(x-t)\, dt
\]

and the natural \( \| \cdot \|_1 \) norm. Sadly this Banach algebra has no unit. It is possible to plug in an identity \( e \) the standard way. Defining a new Banach algebra \( L^1[0, 1] + e \) one can see that there is just one maximal ideal, namely \( L^1[0, 1] \). Consequently \( L^1[0, 1] + e \) has just one multiplicative functional, which is kind of weird!

Definition 79 (Radical). Let \( A \) be a Banach algebra. The intersection of all maximal ideals of \( A \) is called radical of \( A \) and indicated with \( \text{Rad}(A) \).

Proposition 80. Let \( A \) be a Banach algebra. Then

1. \( \text{Rad}(A) = \bigcap_{\alpha \in \Delta} \ker (\alpha) \);
2. \( \text{Rad}(A) = \ker (\Lambda) \).

Proof.

1. Trivial.

2. Remembering the definition of Gelfand transform\(^{11}\), it’s clear that

\[
\text{Rad}(A) \overset{1}{=} \{ x \in A \mid \forall \alpha \in \Delta, \alpha(x) = 0 \} = \{ x \in A \mid \forall \hat{\alpha} \in \Delta, \hat{\alpha} = 0 \} = \ker (\Lambda).
\]

\(^{11}\)Definition 72 on page 20.
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**Theorem 81.** Let $A$ be a Banach algebra and $x \in A$. Then

$$
\|\hat{x}\| = \lim_{n \to +\infty} \sqrt[n]{\|x^n\|_A}
$$

**Proof.** By Theorem 76 on page 21

$$
\|\hat{x}\| = \max_{\alpha \in \Delta} \{|\hat{x}\alpha\} = \max_{\alpha \in \Delta} \{|\alpha(x)\} = \max_{\lambda \in \sigma(x)} \{|\lambda\} =: a
$$

$a$ is the so called *spectral radius*. We want to prove that

$$
a = \lim_{n \to +\infty} \sqrt[n]{\|x^n\|_A}
$$

Let's prove the two inequalities separately.

≥ If $\lambda \in \mathbb{C}$, $|\lambda| < 1$, then $\lambda x - e$ is invertible. Indeed that is true if $\lambda = 0$; if $\lambda \neq 0$

$$
\lambda x - e = \lambda \left( x - \frac{1}{\lambda} e \right)
$$

and $1/|\lambda| > |a|$, so $1/\lambda \in \Omega(x)$. Note that for all $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and $\|\lambda x\| < 1$,

$$
\sum_{k=0}^{+\infty} (\lambda x)^k = \frac{1}{e - \lambda x}
$$

The biggest set on which the function

$$
\lambda \mapsto \sum_{k=0}^{+\infty} (\lambda x)^k
$$

is holomorphic is $B(0,R)$, with

$$
R := \frac{1}{\lim_{n \to +\infty} \sqrt[n]{\|x^n\|}}
$$

(defined the obvious way if the limit is null or infinite), as discussed in Notation 23 on page 10 and Proposition 24 on page 11. On the other hand, by Corollary 27 on page 12, the mapping $\mu \mapsto R_\mu(x) := (x - \mu e)^{-1}$ is holomorphic on its domain. Hence, by equation (1.3.1), the function

$$
\lambda \mapsto \frac{1}{e - \lambda x}
$$
is holomorphic on $B(0, 1/a)$ (defined the obvious way if $a$ is null or infinite). We want to prove that

$$\frac{1}{a} \leq R,$$

i.e. that

$$a \geq \lim_{n \to +\infty} \sqrt[n]{\|x^n\|}.$$

Assume to the contrary that $1/a > R$. It is well known that any holomorphic function $f$ defined on a disk $B(x_0, r)$ concides in the whole disk with its Taylor series $T$ centered at the center of the disk. In other words, if $f$ is holomorphic in $B(x_0, r)$, then its Taylor series $T$ centered at $x_0$ is defined in the whole $B(x_0, r)$ and for all $x \in B(x_0, r)$,

$$f(x) = T(x).$$

In our case, the function (1.3.4) is holomorphic on $B(0, 1/a)$, so its Taylor series, which by (1.3.2) is (1.3.3), should converge on the whole $B(0, 1/a)$. Sadly we supposed that it’s radius of convergence $R$ were strictly lower than $1/a$. Absurd.

\[\square\]

**Corollary 82.** Let $A$ be a Banach algebra. Then $x \in \text{Rad}(A)$ if and only if, for $n \to +\infty$,

$$\sqrt[n]{\|x\|^n} \to 0.$$

**Remark 83.** Clearly nilpotent elements of $A$ belong to $\text{Rad}(A)$. For this reason elements of $\text{Rad}(A)$ are sometimes called generalized nilpotents. Note that, belonging to $\text{Rad}(A)$, every nilpotent is annihilated by all $\alpha \in \Delta$.

**Definition 84** (Semisimple algebra). Let $A$ be a Banach algebra. $A$ is called semisimple if $\text{Rad}(A) = \{0\}$.

**Proposition 85.** Let $A$ be a semisimple Banach algebra. Then $\Lambda$ is injective.

**Proof.** Trivial. \[\square\]

**Notation 86.** Let $A$ be a Banach algebra. The following notation will be used throughout the paper:

$$\hat{A} := \Lambda(A).$$

**Corollary 87.** Let $A$ be a semisimple Banach algebra. Then $\hat{A}$ is a linear isomorphism onto $\hat{A} \subset C(\Delta; \mathbb{C})$.

**Proof.** Trivial. \[\square\]

**Proposition 88.** Let $A$ be a Banach algebra. Then $\hat{A}$ is an algebra with respect to the pointwise sum and product.
Corollary 89. Let \( A \) be a semisimple Banach algebra. Then \( \hat{A} \) is a Banach algebra with respect to the pointwise sum and product.

Proof. Trivial. \( \square \)

Definition 90 (Total set). Let \( V \) be a linear space and \( x \in V \). We say that \( D \subseteq V^* \) is total if
\[
\forall \alpha \in D, \ \alpha(x) = 0 \iff x = 0.
\]

Proposition 91. Let \( A \) be a Banach algebra. Then \( A \) is semisimple if and only if \( \Delta \) is total.

Proof. Suppose \( A \) semisimple, then \( \ker(A) = 0 \), hence the only \( x \in A \) such that \( \hat{x} \equiv 0 \) is \( x = 0 \). If \( x \in A \) and for all \( \alpha \in \Delta \)
\[
0 = \alpha(x) = \hat{x} \alpha,
\]
then \( \hat{x} \equiv 0 \), hence \( x = 0 \); if \( x = 0 \), for all \( \alpha \in \Delta \)
\[
\alpha(0) = 0.
\]

Vice versa, suppose that \( \Delta \) is total. Assume to the contrary that \( A \) is not semisimple, i.e. that there exists \( x \in A \setminus \{0\} \) such that \( \hat{x} \equiv 0 \). By definition, it follows that for all \( \alpha \in \Delta \),
\[
0 = \hat{x} \alpha = \alpha(x).
\]
Being \( x \neq 0 \), this implies that \( \Delta \) is not total. Absurd. \( \square \)

Corollary 92 (Of Theorem 81). Let \( A \) be a Banach algebra such that, for all \( x \in A \),
\[
\|x^2\| = \|x\|^2. \tag{1.3.5}
\]
Then \( \Lambda: A \to \mathbb{C}(\Delta; \mathbb{C}) \) is an isometry.

Proof. Since the first equation holds and the first limit exists
\[
\| \hat{x} \| = \lim_{n \to +\infty} \sqrt[n]{\|x^n\|} = \lim_{n \to +\infty} 2^n \sqrt[n]{\|x^{2^n}\|} = \lim_{n \to +\infty} 2^n \sqrt[n]{\|(x^{2^{n-1}})^2\|} = \lim_{n \to +\infty} 2^n \sqrt[n]{\|x^{2^{n-1}}\|^2} = \ldots = \lim_{n \to +\infty} 2^n \sqrt[n]{\|x\|^{2^n}} = \|x\|. \tag*{\square}
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Corollary 93. Let $A$ be a Banach algebra. If (1.3.5) holds, then $\hat{A}$ is complete.

Proof. It follows directly from the previous result. \qed

Remark 94. The (1.3.5) is a strong request, which is not trivially satisfied by all Banach algebras. $C[0,1]$ satisfies it, but operators algebras, where the product is the composition, usually do not.

Definition 95 (Symmetric space). Let $A$ be a Banach algebra. $\hat{A}$ is called symmetric if for all $x \in A$ there is $y \in A$ such that for all $\alpha \in \Delta$

$$\alpha(x) = \alpha(y)$$

Example 96. $\hat{C}[0,1]$ is symmetric. Indeed multiplicative functionals of $C[0,1]$ are Dirac deltas, so for all $x \in C[0,1], y = \pi$ satisfies the property.

Theorem 97 (Stone-Weierstrass). Let $K$ be a compact $T_2$ topological space, $X \subset C(K;\mathbb{C})$ a closed subspace and assume that:

1. the constant function $1$ belongs to $X$;
2. $X$ separates the points of $K$, i.e. for all $t_1, t_2 \in K, t_1 \neq t_2$, there exists $f \in X$ such that $f(t_1) \neq f(t_2)$;
3. for all $f, g \in X$, the pointwise product $fg \in X$;
4. for all $g \in X, \bar{g} \in X$.

Then $X = C(K;\mathbb{C})$.

Theorem 98. Let $A$ be a Banach algebra. Assume that $\hat{A}$ is symmetric and for all $x \in A, \|x^2\| = \|x\|^2$. Then $\hat{A} = C(\Delta;\mathbb{C})$, i.e. $A$ and $C(\Delta;\mathbb{C})$ are isometric via $\Lambda$.

Proof. Let’s apply Stone-Weierstrass theorem for $X = \hat{A} \subset C(\Delta;\mathbb{C})$. Since (1.3.5) holds, $\hat{A}$ is complete, hence closed. By Proposition on page 24, $\hat{A}$ is a linear space. Let’s check all other conditions:

1. the constant function $1$ belongs to $\hat{A}$, indeed

$$\Lambda e \equiv \hat{\varepsilon} \equiv [\alpha \mapsto \alpha(e) = 1] \equiv 1;$$

2. $\hat{A}$ separates points of $\Delta$, indeed if $\alpha, \beta \in \Delta$ and for all $\hat{x} \in \hat{A}$

$$\alpha(x) = \hat{x}(\alpha) = \hat{x}(\beta) = \beta(x),$$

then $\alpha = \beta$;

3. by Proposition on page 24, $\hat{A}$ is an algebra with respect to the pointwise product;
4. fix $\tilde{x} \in \tilde{A}$; since $\tilde{A}$ is symmetric there is $y \in X$ such that, for all $\alpha \in \Delta$,

$$\tilde{x}(\alpha) = \overline{\alpha(x)} = \alpha(y) = \tilde{y}(\alpha),$$

hence $\tilde{x} = \tilde{y} \in \tilde{A}$.

\[\square\]

**Example 99.** We stress again that the hypothesis aren’t trivial. The Banach algebra of functions continue on the closed disk and holomorphic in the interior doesn’t satisfy them. Same goes for the Banach algebra of functions which are sum of Fourier series.

### 1.4 Involutions

**Definition 100** (Involution). Let $A$ be a Banach algebra. A mapping

$$A \rightarrow A,$$

$$x \mapsto x^*,$$

such that

1. for all $x, y \in A$,

$$(x + y)^* = x^* + y^*;$$

2. for all $x \in A$, for all $\lambda \in \mathbb{C}$,

$$(\lambda x)^* = \overline{\lambda} x^*;$$

3. for all $x, y \in A$,

$$(xy)^* = y^* x^*;$$

4. for all $x \in A$,

$$(x^*)^* = x$$

is called involution. If such a mapping is defined on $A$, $A$ is called an algebra with involution.

**Remark 101.** From now on we will always assume that the Banach algebras we consider are algebras with involutions.

**Example 102.** In $\mathbb{C} [0, 1]$ we define the involution as the conjugate map:

$$\forall g \in \mathbb{C} [0, 1], \ g^* = \overline{g}.$$

**Remark 103.** Even if we work with commutative Banach algebras, we wrote the third property of an involution in such a way that makes sense even for non commutative ones.
1.4 Involutions

Exercise 104. Let $H$ be a Hilbert space and consider the Banach algebra of bounded linear operator $A = BL(H; H)$. This is an important example of a non commutative Banach algebra. Here one can get an involution via the dual operator

$$BL(H; H) \mapsto BL(H; H),$$

$$D \mapsto D^*,$$

where $D^*$ is the unique operator such that, for all $x, y \in H$

$$\langle D^* x, y \rangle = \langle x, Dy \rangle.$$

We will study some commutative subalgebras of $B(H; H)$.

Definition 105 (Self-adjoint element). Let $A$ be a Banach algebra and $x \in A$. If $x^* = x$ we say that $x$ is self-adjoint.

Exercise 106. In $C[0, 1]$ all real functions are self-adjoint.

Proposition 107. Let $A$ be a Banach algebra. Then

1. for all $x \in A$, $(x + x^*), i(x - x^*)$ and $xx^*$ are self-adjoint;
2. for all $x \in A$, exist two unique self-adjoint $u, v \in A$ such that $x = u + iv$;
3. $e^* = e$;
4. for all $x \in A$, $x$ is invertible if and only if $x^*$ is invertible and

$$\sigma(x) = \overline{\sigma(x^*)}.$$  

5. for all $x \in A$,

$$\sigma(x) = \overline{\sigma(x^*)},$$

where $\overline{\sigma(x^*)}$ denotes the complex conjugate of $\sigma(x^*)$.

Proof.

1. Trivial.

2. Fix $x \in A$. Clearly

$$u = \frac{1}{2} (x + x^*) \quad \text{and} \quad v = -i \frac{1}{2} (x - x^*)$$

do the trick. For the uniqueness take another $u', v' \in A$ self-adjoint and such that $x = u' + iv'$. Then

$$u' - iv' = x^* = u - iv.$$

Hence

$$2w = x + x^* = 2u$$

and this easily brings us the uniqueness.
3. Note that
\[ e = (e^*)^* = (e^*e)^* = e^*e^{**} = e. \]

4. For all \( x \in A \),
\[ e = (xx^{-1})^* = x^*(x^{-1})^*. \]

5. Fix any \( x \in A \). Take any \( \lambda \notin \sigma(x) \). Then \( x - \lambda e \) is invertible. By the previous point also \((x - \lambda e)^*\) is invertible and
\[ ((x - \lambda e)^*)^{-1} = ((x - \lambda e)^{-1})^*. \]

Since
\[ (x - \lambda e)^* = x^* - \lambda e, \]
\[ \lambda \notin \sigma(x^*) \] or equivalently \( \lambda \notin \sigma(x^*) \). Clearly the converse holds too, i.e. \( \lambda \notin \sigma(x) \) if and only if \( \lambda \notin \sigma(x^*) \).

\[ \square \]

**Remark 108.** Note that self-adjoints elements play the same role that real numbers play in the field of complex number.

**Remark 109.** Remember that a Banach algebra \( A \) is semisimple if and only if \( \Delta \) is total, i.e. for all \( x \in A \setminus \{0\} \) there exists \( h \in \Delta \) such that \( h(x) \neq 0 \).

**Lemma 110.** Let \( A \) be a semisimple Banach algebra and \( h \in \Delta \). Then, the functional
\[ \varphi : A \rightarrow \mathbb{C}, \quad x \mapsto \overline{h(x^*)} \]
is multiplicative (i.e. \( \varphi \in \Delta \)).

**Proof.** For all \( x, y \in A \),
\[ \varphi(x + y) = \overline{h((x + y)^*)} \]
\[ = \overline{h(x^* + y^*)} \]
\[ = \overline{h(x^*) + h(y^*)} \]
\[ = \varphi(x) + \varphi(y). \]
Clearly replacing the sum with the multiplication in \( A \), the same equalities hold. For all \( \lambda \in \mathbb{C} \), for all \( x \in A \)
\[ \varphi(\lambda x) = \overline{h((\lambda x)^*)} \]
\[ = \overline{h(\lambda x^*)} \]
\[ = \overline{\lambda h(x^*)} \]
\[ = \lambda \varphi(x). \]
Finally, \( \varphi \neq 0 \), indeed \( \varphi(e) \equiv 1 \). \[ \square \]
Corollary 111. Let $A$ be a semisimple Banach algebra and $h \in \Delta$. Then, the functional

$$\varphi: A \to \mathbb{C}, \quad x \mapsto h(x^*)$$

is continuous.

Proposition 112. Let $A$ be a semisimple Banach algebra. Then the involution is continuous, i.e. if $\{x_n\}_{n \in \mathbb{N}} \subset A$, $x \in A$ and $x_n \xrightarrow{n \to +\infty} x$, then

$$x_n^* \xrightarrow{n \to +\infty} x^*.$$

Proof. We use the Closed Graph Theorem\(^{12}\). Take the convolution as an anti-linear operator on $A$. The theorem states that, if

$$\left[ x_n \xrightarrow{n \to +\infty} x, x_n^* \xrightarrow{n \to +\infty} y \implies y = x^* \right],$$

then the involution operator is continuous. Let’s check the premises. Fix any $h \in \Delta$. If $x_n \xrightarrow{n \to +\infty} x$ and $x_n^* \xrightarrow{n \to +\infty} y$, by the previous corollary

$$h(x^*) = \varphi(x) = \lim_{n \to +\infty} \varphi(x_n) = \lim_{n \to +\infty} h(x_n^*) = h(y),$$

hence

$$h(x^*) = h(y) \iff h(x^* - y) = 0.$$

Being $A$ semisimple, $\Delta$ is total. Being $h$ arbitrary, it follows $x^* = y$. \(\square\)

Definition 113 ($B^*$-algebra). Let $A$ be Banach algebra. $A$ is called a $B^*$-algebra if $A$ is with involution and for all $x \in A$

$$\|x \cdot x^*\| = \|x\|^2.$$

In non commutative cases those algebras are also known as $C^*$-algebras.

Exercise 114. $C[0, 1]$ is a $B^*$-algebra with respect to the involution defined for all $f \in C[0, 1]$ by the complex conjugate

$$f^* = \overline{f}.$$

Lemma 115. Let $A$ be a $B^*$-algebra, $u \in A$ self-adjoint and $h \in \Delta$. Then $h(u) \in \mathbb{R}$.

---

\(^{12}\)The Closed Graph Theorem is usually stated for linear operators. Check that the same theorem holds for anti-linear operators, i.e. operators that are additive and anti-homogeneous, like the involution.
Proof. Fix $\alpha, \beta \in \mathbb{R}$ such that $h(u) = \alpha + i\beta$. For all $t \in \mathbb{R}$, take

$$z := u + ite.$$  

Then, for all $t \in \mathbb{R}$

$$z^* = u - ite,$$
$$zz^* = u^2 + t^2e,$$
$$|h(z)|^2 \leq ||z||^2 = ||zz^*|| \leq ||u^2|| + t^2$$

but also

$$|h(z)|^2 = |h(u) + it|^2 = |\alpha + i(\beta + t)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2.$$

Since, for all $t \in \mathbb{R}$

$$\alpha^2 + \beta^2 + 2\beta t + t^2 \leq ||u^2|| + t^2$$

This clearly implies that $\beta = 0$ (the right hand side is independent of $t$). \hfill \Box

Corollary 116. Let $A$ be a $B^*$-algebra and $u \in A$ self-adjoint. Then $\hat{u} \in \hat{A}$ is a real functional.

Theorem 117 (Gelfand-Naimark). Let $A$ be a $B^*$-algebra. Then $A$ is isometric to $\mathcal{C}(\Delta; \mathbb{C})$ via the Gelfand transform. Furthermore, for all $x \in A$

$$\Lambda(x^*) = \overline{\Lambda(x)} \quad (1.4.1)$$

i.e. for all $h \in \Delta$,

$$h(x^*) = \overline{h(x)}. \quad (1.4.2)$$

Proof. Let’s prove that $\hat{A} = \mathcal{C}(\Delta; \mathbb{C})$ via the Stone-Weierstrass theorem:

1. the constant function $1 = \hat{1}$ belongs to $\hat{A}$;

2. $\hat{A}$ separates the points of $\Delta$, i.e. for all $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, there exists $\hat{x} \in \hat{A}$ such that $\alpha(x) = \hat{x}(\alpha) \neq \hat{x}(\beta) = \beta(x)$;

i.e. $\alpha = \beta$, absurd;
3. for all \( \hat{x}, \hat{y} \in \hat{A} \), the pointwise product \( \hat{x}\hat{y} \in \hat{A} \);
   for all \( \alpha \in \Delta \),
   \[ (\hat{x}\hat{y})(\alpha) = \hat{x}(\alpha)\hat{y}(\alpha) \]
   \[ = \alpha(x)\alpha(y) = \alpha(xy) = [\Lambda(xy)]\alpha; \]
4. for all \( \hat{x} \in \hat{A}, \hat{x} \in \hat{A} \);
   fix any \( x \in A \) and \( u, v \in A \) self-adjoint such that \( x = u + iv \); then \( x^* = u - iv \) and \( \hat{x} = \hat{u} + i\hat{v} \); by previous corollary
   \[ \Lambda(x^*) = \hat{u} - i\hat{v} = \hat{x} \]
   which also proves formulae (1.4.1) and (1.4.2).

We now want to prove that \( \Lambda: A \rightarrow \hat{A} \) is an isometry. If we do that, we prove that \( \hat{A} \) is closed in \( C(\Delta; \mathbb{C}) \), by Stone-Weierstrass theorem we can conclude that \( \hat{A} = C(\Delta; \mathbb{C}) \) and consequently that \( A \) and \( C(\Delta; \mathbb{C}) \) are isometric via \( \Lambda \).

- Take any \( y \in A \) self-adjoint. Then
  \[ \|y\|^2 = \|yy^*\| = \|y\|^2. \]
  Clearly, for all \( n \in \mathbb{N} \), also \( y^{2n} \) is self-adjoint and
  \[ \|y^{2n}\| = \|y\|^{2^n}. \]
  By Theorem 81 on page 23,
  \[ \|\hat{y}\|_{\hat{A}} = \lim_{n \to +\infty} \sqrt[n]{\|y^n\|} \]
  \[ = \lim_{n \to +\infty} \sqrt[n]{\|y^{2n}\|} \]
  \[ = \lim_{n \to +\infty} \sqrt[n]{\|y\|^{2^n}} \]
  \[ = \|y\|. \]
- Take any \( x \in A \) and call \( y := xx^* \). Note that \( y \) is self-adjoint. Then
  \[ \|x\|^2 = \|xx^*\| = \|y\| = \|\hat{y}\|_{\hat{A}} = \|\hat{x}\Lambda(x^*)\|_{\hat{A}} = \|\hat{x}\|_{\hat{A}}^2 = \|\hat{x}\|^2_{\hat{A}}. \]

\( \Box \)

1.5 Applications

1.5.1 Wiener algebra

**Example 118.** The so called *Wiener algebra* \( W \) is the set of \( 2\pi \)-periodic complex functions which are absolute convergent sum of their Fourier series

\[ \left\{ f: [0, 2\pi) \rightarrow \mathbb{C} \ | \ \exists \ \{c_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}, \sum_{k \in \mathbb{Z}} |c_n| < +\infty, \forall t \in [0, 2\pi), \ f(t) = \sum_{k \in \mathbb{Z}} c_ne^{ikt} \right\} \]
endowed with the pointwise sum, pointwise product and the norm defined for all \( f \in W \) such that \( f = \sum_{k \in \mathbb{Z}} c_k e^{ik(\cdot)} \) by
\[
\|f\| = \sum_{k \in \mathbb{Z}} |c_k|.
\]
The function \( f \equiv 1 \in W \) is the unit of \( W \). It’s easy to check that \( W \) is Banach algebra\(^{13}\).

**Lemma 119.** Let \( W \) be the Wiener algebra. Then
\[
\Delta = \{ \alpha_{t_0} := \{ f \mapsto f(t_0) \} \}_{t_0 \in [0,2\pi)}.
\]

**Proof.** Clearly
\[
\Delta \supset \{ \alpha_{t_0} \}_{t_0 \in [0,2\pi)},
\]
indeed for all \( t_0 \in [0,2\pi) \) and for all \( f, g \in W \)
\[
\alpha_{t_0} (fg) = \alpha_{t_0} (f(t_0)g(t_0)) = \alpha_{t_0} (f) \alpha_{t_0} (g).
\]
Let’s prove the converse. Fix any \( \alpha \in \Delta \). Note that \( e^{i(\cdot)} \in W \) and call
\[
a := \alpha \left( e^{i(\cdot)} \right).
\]
Remember\(^{14}\) that
\[
\alpha \left( e^{-i(\cdot)} \right) = \left( \alpha \left( e^{i(\cdot)} \right) \right)^{-1} = \frac{1}{a}
\]
and that multiplicative functional has norm 1, hence
\[
|a| = \left| \alpha \left( e^{i(\cdot)} \right) \right| \leq \left\| e^{i(\cdot)} \right\| = 1,
\]
\[
1/|a| = \left| \alpha \left( e^{-i(\cdot)} \right) \right| \leq \left\| e^{-i(\cdot)} \right\| = 1,
\]
and consequently \( |a| = 1 \). This means that there exists \( t_0 \in [0,2\pi) \) such that
\[
a = e^{it_0}.
\]
We have proved that
\[
\alpha \left( e^{i(\cdot)} \right) = e^{it_0}.
\]
Being \( \alpha \) multiplicative, for all \( n \in \mathbb{Z} \),
\[
\alpha \left( e^{in(\cdot)} \right) = e^{int_0}.
\]
This means that for all \( \{c_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{C} \) such that \( \sum_{n \in \mathbb{Z}} |c_n| < +\infty \),
\[
\alpha \left( \sum_{n \in \mathbb{Z}} c_n e^{in(\cdot)} \right) = \sum_{n \in \mathbb{Z}} c_n e^{int_0}.
\]
\( \square \)

\(^{13}\)For the completeness, note that \( W \) is isomorphic to \( \ell^1(\mathbb{C}) \).

\(^{14}\)See Proposition 48 on page 36.
1.5 Applications

Theorem 120 (Wiener). Let \( W \) be the Wiener algebra. If \( f \in W \) and for all \( t \in [0, 2\pi) \), \( f \neq 0 \), then
\[
\frac{1}{f} \in W.
\]

*Proof.* By previous exercise and Corollary 64 on page 19 it follows that for any maximal ideal \( M \subset W \), \( f \notin M \). By Proposition 44 on page 16, \( 1/f \in W \). \(\square\)

1.5.2 Disk algebra

**Exercise 121.** The so called disk algebra \( A \) is the set of complex continuous functions defined on the closed complex disk whose restriction to the interior of the disk is holomorphic
\[
A := \{ f \in \mathcal{C}(\overline{D}; \mathbb{C}) \mid f|_{D} \text{ is holomorphic} \}
\]
endowed with the pointwise sum, pointwise product and the norm defined for all \( f \in A \) by
\[
\|f\| = \max_{z \in D} |f(z)| = \max_{z \in \partial D} |f(z)|
\]
(last identity holds because of the maximum modulus principle). The function \( f \equiv 1 \in A \) is the unit of \( A \). It’s easy to check that \( A \) is Banach algebra.

**Proposition 122.** Let \( A \) be the disk algebra. Then
\[
\Delta = \{ \alpha_{z_{0}} := \{ f \mapsto f(z_{0}) \} \}_{z_{0} \in \overline{D}}.
\]

*Proof.* Clearly
\[
\Delta \supset \{ \alpha_{z_{0}} \}_{z_{0} \in \overline{D}}.
\]

Viceversa, fix any \( \alpha \in \Delta \). We want to prove that there exists \( z_{0} \in \overline{D} \) such that \( \alpha = \alpha_{z_{0}} \). Call
\[
id: D \rightarrow \overline{D},
\]
\[
z \mapsto z
\]
and
\[
\alpha \circ (\text{id}) =: z_{0} \in \mathbb{C}.
\]
Note that \( z_{0} \in \overline{D} \), indeed
\[
|\alpha (g)| \leq \|\text{id}\| = 1.
\]

*Claim.* For all \( p \in A \) polynomial, \( \alpha (p) = \alpha_{z_{0}}(p) \).

*Proof.* Fix any \( p \in A \) polynomial. Then there exist \( \{a_{k}\}_{k=0}^{n} \subset \mathbb{C} \) such that
\[
p = a_{n}\text{id}^{n} + a_{n-1}\text{id}^{n-1} + \ldots + a_{1}\text{id} + a_{0}
\]
and \( \alpha \) is linear and multiplicative. \(\square\)
1.5 Applications

Claim. Polynomials are dense in $A^{15}$.

Proof. Fix $f \in A$ and $\varepsilon > 0$. We want to prove that there exist a polynomial $p$ such that $\|f - p\| < \varepsilon$. Being $f$ continuous on a compact set, by Heine-Cantor theorem its uniformly continuous. Then there is $\delta \in (0, 1)$ such that, for all $z_1, z_2 \in D$ with $|z_1 - z_2| < \delta$,

$$|f(z_1) - f(z_2)| < \frac{\varepsilon}{2}.$$  

Fix that $\delta$. Note that the function

$$\tilde{f}: (1 + \delta)D \to \mathbb{C},$$

$$z \mapsto \tilde{f}(z) := f\left(\frac{z}{1 + \delta}\right)$$

is holomorphic on $(1 + \delta)D \supset D$.

Hence $\tilde{f}$ Taylor series centered at 0 uniformly converges to $\tilde{f}$ on compact subsets included in $(1 + \delta)D$ (and including 0). So there exists a polynomial $p$ such that

$$\sup_{z \in D} \left| f\left(\frac{z}{1 + \delta}\right) - p(z) \right| < \frac{\varepsilon}{2}.$$ 

Note that for all $z \in D$

$$\left| z - \frac{z}{1 + \delta} \right| = \delta \frac{|z|}{1 + \delta} \leq \frac{\delta}{1 + \delta} < \frac{\delta}{1 + 1} = \delta.$$ 

Consequently, for all $z \in D$,

$$|f(z) - p(z)| = \left| f(z) - f\left(\frac{z}{1 + \eta}\right) + f\left(\frac{z}{1 + \eta}\right) - p(z) \right| \leq \left| f(z) - f\left(\frac{z}{1 + \eta}\right) \right| + \left| f\left(\frac{z}{1 + \eta}\right) - p(z) \right| \leq \varepsilon.$$ 

\[\Box\]

\[\text{Note that this doesn’t follow from Stone-Weierstrass theorem (as in the real case) because the algebra of complex polynomials is not closed under complex conjugation.}\]
1.5 Applications

Being both \(\alpha\) and \(\alpha_{z_0}\) continuous the result easily follows from density. Indeed, if \(\{p_n\}_{n \in \mathbb{N}}\) is a sequence of polynomials such that, for \(n \to +\infty\), \(p_n \to f\), then

\[
\alpha(f) = \alpha\left(\lim_{n \to +\infty} p_n\right) = \lim_{n \to +\infty} \alpha(p_n) = \lim_{n \to +\infty} \alpha_{z_0}(p_n) = \alpha_{z_0}\left(\lim_{n \to +\infty} p_n\right) = \alpha_{z_0}(f).
\]

\(\square\)

**Theorem 123.** Let \(A\) be the disk algebra and \(f_1, \ldots, f_m \in A\) such that for all \(z \in \overline{D}\) there is \(j \in \{1, \ldots, m\}\) such that \(f_j(z) \neq 0\). Then there is \(h_1, \ldots, h_m \in A\) such that

\[
\sum_{k=1}^{m} h_k f_k \equiv 1.
\]

**Proof.** Consider the set

\[
I = \left\{ g = \sum_{k=1}^{m} g_k f_k \left| g_1, \ldots, g_m \in A \right. \right\}.
\]

Clearly \(I\) is a subspace of \(A\), is different from \(\{0\}\) and is closed under the product with elements in \(A\). We want to prove that \(I = A\). If this is true, then \(1 \in I\) and the theorem is proved. Assume to the contrary that \(I \subsetneq A\). Then \(I\) is an ideal (by definition). By Theorem 43 on page 15 there is \(M\) maximal ideal such that \(M \supset I\). By previous proposition and Corollary 64 on page 19 there exists \(z_0 \in \overline{D}\) such that

\[
M = \{ f \in A \left| f(z_0) = 0 \right. \}.
\]

Hence all functions in \(I\) vanishes in \(z_0\). For all \(j \in \{1, \ldots, m\}\) take \(g_j = 1\) and \(g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_m = 0\). It follows that for all \(j \in \{1, \ldots, m\}\), \(f_j(z_0) = 0\). Absurd. \(\square\)

1.5.3 Complex compact operators in Hilbert spaces

**Definition 124** (Compact operator). Let \(H\) be a Hilbert space. An operator \(T: H \to H\) is called **compact** if \(T(B_H)\) is relatively compact.

**Definition 125** (Self-adjoint operator). Let \(H\) be a Hilbert space and \(T: H \to H\) a linear operator. \(T\) is called **self-adjoint** if for all \(x, y \in H\)

\[
\langle Tx, y \rangle = \langle x, Ty \rangle.
\]
Exercise 126. Let $H$ be a Hilbert space and $T : H \to H$ a compact self-adjoint operator. Consider the set of polynomials in $T$

$$\mathcal{P}_T := \{a_nT^n + a_{n-1}T^{n-1} + \ldots + a_0 \mid n \in \mathbb{N}_0, a_0, \ldots, a_n \in \mathbb{C}\}.$$ 

The set $\mathcal{P}_T$, endowed with the obvious sum and scalar multiplication is a normed algebra (check it) with respect to the norm defined for all $P \in \mathcal{P}_T$ by

$$\|P\| = \sup_{\|x\| \leq 1} \|Px\|.$$  

$\mathcal{P}_T$ is an incomplete normed algebra (check it). Denote with $A$ the completion of $\mathcal{P}_T$. By definition polynomials in $T$ are dense in $A$. Clearly $A$ has a unit. $A$ also has an involution$^{16}$. Furthermore, $A$ is a $B^*$-algebra. Let’s check it for $p = T$. Note that (check it), for all $B : H \to H$ self-adjoint,

$$\|B\| = \sup_{\|x\| \leq 1} |\langle Bx, x \rangle|.$$ 

Note that (check it) if $T$ is self-adjoint, also $T^2$ is self-adjoint. Hence

$$\|T^2\| = \sup_{\|x\| \leq 1} |\langle T^2x, x \rangle|$$

$$= \sup_{\|x\| \leq 1} |\langle Tx, Tx \rangle|$$

$$= \sup_{\|x\| \leq 1} \|Tx\|^2$$

$$= \|T\|^2.$$ 

The same equalities holds for all $X \in A$ (check it).

Remark 127. By Gelfand-Naimark theorem$^{17}$, since $A$ is a $B^*$-algebra, $A$ is isomorphic to $C(\Delta; \mathbb{C})$ via the Gelfand transform. We want to say explicitly who is $C(\Delta, \mathbb{C})$. In order to do that, we need the spectral theorem for compact self-adjoint operators.

Definition 128 (Eigenvalues, eigenvectors, spectrum and eigenspaces). Let $H$ be a Hilbert space and $T : H \to H$ a linear operator. We say that $\lambda \in \mathbb{C}$ is an eigenvalue for $T$ if there exists $x \in H \setminus \{0\}$, called eigenvector, such that $Tx = \lambda x$. The set of all eigenvalues is called spectrum. Let $\lambda$ be an eigenvalue for $T$, then the linear space $E_\lambda := \{x \in H \mid Tx = \lambda x\}$ is called eigenspace (relative to $\lambda$).

Proposition 129. Let $H$ be a Hilbert space and $T$ a compact self-adjoint operator. Then

$^{16}$Which doesn’t do much since $T^* = T$.

$^{17}$Theorem 117 on page 31.
1.5 Applications

1. $T$ has has most contably many eigenvalues;
2. all eigenvalues are reals;
3. all eigenspaces are finite-dimensional.

**Theorem 130** (Spectral theorem for compact self-adjoint operators). Let $H$ be a Hilbert space and $T$ a compact self-adjoint operator. Assume that the spectrum is countable\(^\text{18}\) and $\ker(T) = \{0\}^{19}$. If $\sigma := \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ is the spectrum, for all $\lambda \in \sigma$ we denote with $E_\lambda$ the eigenspace relative to $\lambda$ and with $P_\lambda : H \to E_\lambda$ the orthogonal projector on the eigenspace. Assuming without loss of generality that $|\lambda_1| > |\lambda_2| > \ldots$, it follows

$$T = \sum_{i=1}^{+\infty} \lambda_i P_\lambda.$$ 

**Remark 131.** To describe the set $\Delta$ of multiplicative functionals of $A$ we should think about the previous result in a weak sense. Fix any $x \in H$. Then

$$Tx = \sum_{i=1}^{+\infty} \lambda_i P_\lambda x$$

and this series is convergent in the norm of the Hilbert space\(^{20}\). By Theorem 76 on page 21 we have

$$\sigma(x) = \{\alpha(x)\}_{\alpha \in \Delta}.$$ 

Since, by previous theorem, the spectrum is countable, also $\Delta$ is countable, say

$$\Delta = : \{\alpha_0, \alpha_1, \alpha_2, \ldots\}.$$ 

Since $\mathcal{P}_T$ is dense in $A$, to describe each element of $\Delta$ it's enough to say how it behaves on polynomials in $T$. Without loss of generality, for all $P \in \mathcal{P}_T$ and for all $i \in \mathbb{N}$

$$\alpha_i(P) = P(\lambda_i)$$

and, if $P := a_n T^n + \ldots + a_0 I$,

$$\alpha_0(P) = a_0 (a_n T^n + \ldots + a_0 I) = a_0.$$ 

Checking that everything makes sense it is possible to conclude that

$$\Delta = c.$$

---

\(^{18}\)The finite version should be well-known from previous courses.

\(^{19}\)This hypothesis is unnecessary and only assumed for the sake of simplicity. It should be clear how the results would change if this hypothesis was dropped.

\(^{20}\)If you think about finite-dimentional case, the well-known spectral theorem says that if a matrix is symmetric (which is the equivalent of being self-adjoint) there is a basis that makes it diagonal. This is a generalization of the same concept. Indeed the previous identity defines a diagonal matrix.
Chapter 2

Seminars

2.1 Gelfand and Fourier transform (Tommaso Cesari)

Proposition 132. \((L^1(\mathbb{R}^n), \| \cdot \|_1)\) is a Banach algebra without unit with respect to the convolution.

Proof. It should be well known from previous courses. \qed

Remark 133. Proposition 4 on page 4 states that any Banach algebra can be enlarged enough to admit a unit element. Note that in the particular case of \(L^1(\mathbb{R}^n)\), the complex component added to \(L^1(\mathbb{R}^n)\) in order to get the unit can be thought as a coefficient of a function \(\delta\), defined for all \(x \in \mathbb{R}^n\) as the Dirac measure on \(\mathbb{R}^n\). Indeed the set

\[ \mathcal{L} := \{ f + \alpha \delta \mid f \in L^1(\mathbb{R}^n), \alpha \in \mathbb{C} \} \]

is a Banach algebra with unit \(\delta\) with respect to the operations

\[ \forall f + \alpha \delta, g + \beta \delta \in \mathcal{L}, \quad (f + \alpha \delta) + (g + \beta \delta) := (f + g) + (\alpha + \beta) \delta, \]
\[ \forall f + \alpha \delta, \forall \beta \in \mathbb{C}, \quad \beta (f + \alpha \delta) := \beta f + (\alpha \beta) \delta, \]
\[ \forall f + \alpha \delta, g + \beta \delta \in \mathcal{L}, \quad (f + \alpha \delta) \ast (g + \beta \delta) := (f \ast g + \beta f + \alpha g) + (\alpha \beta) \delta, \]

the norm

\[ \forall f + \alpha \delta \in \mathcal{L}, \quad \| f + \alpha \delta \| = \| f \|_1 + |\alpha| \]

and \(L^1(\mathbb{R}^n)\) is isometrically embedded in \(\mathcal{L}\) via \(f \mapsto f + 0 \cdot \delta\). Note that, for all \(x \in \mathbb{R}^n\),

\[ (f \ast \delta) (x) = \int_{\mathbb{R}^n} f(t) d\delta_x(t), \]

so the convolution \(f \ast \delta\) actually coincide with the integration of \(f\) with respect to the Dirac measure.
Notation 134. We fix the notation $\mathcal{L}$ to indicate the smallest Banach algebra with unit containing $(L^1(\mathbb{R}^n);*)$, as described in the previous remark.

Notation 135 (Fourier transform). For all $f \in L^1(\mathbb{R}^n)$, $\hat{f}$ and $\mathcal{F}(f)$ will denote the Fourier transform of $f$.

Proposition 136. The function
\[ h: \mathbb{R}^n \rightarrow \Delta, \]
\[ t \mapsto h_t := \begin{cases} [f + \alpha \delta \mapsto \hat{f}(t) + \alpha], & \text{if } t \in \mathbb{R}^n, \\ [f + \alpha \delta \mapsto \alpha], & \text{if } t = \infty \end{cases} \]
is a homeomorphism between $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$ with the topology of the one-point compactification of $\mathbb{R}^n$ and the set $\Delta$ of multiplicative functional on $\mathcal{L}$ with the usual Gelfand topology. I.e., as topological spaces
\[ \Delta = \mathbb{R}^n. \]

Proof. For all $f \in L^1(\mathbb{R}^n)$, define
\[ \hat{f}(\infty) := \lim_{t \rightarrow \infty} \hat{f}(t) = 0. \]
Let’s start by proving that $h$ is well defined. For all $(f + \alpha \delta), (g + \beta \delta) \in \mathcal{L}$ and for all $t \in \mathbb{R}^n$
\[ h_t ((f + \alpha \delta) * (g + \beta \delta)) = h_t ((f * g + \beta f + \alpha g) + (\alpha \beta) \delta) \]
\[ = \mathcal{F}(f * g + \beta f + \alpha g) + \alpha \beta \]
\[ = \mathcal{F}(f * g) + \mathcal{F}(\beta f) + \mathcal{F}(\alpha g) + \alpha \beta \]
\[ = \hat{f}(t) \hat{g}(t) + \beta \hat{f}(t) + \alpha \hat{g}(t) + \alpha \beta \]
\[ = \left( \hat{f}(t) + \alpha \right) \left( \hat{g}(t) + \beta \right) \]
\[ = h_t (f + \alpha \delta) h_t (f + \beta \delta). \]
Now we prove that $h$ is injective. If $t_1, t_2 \in \mathbb{R}^n$, $t_1 \neq t_2$, assume (by contradiction)
\[ [f + \alpha \delta \mapsto \hat{f}(t_1) + \alpha] = [g + \beta \delta \mapsto \hat{g}(t_2) + \beta], \]
this means in particular that for all $f \in L^1(\mathbb{R}^n)$
\[ \hat{f}(t_1) = \hat{f}(t_2). \]
Of course this cannot happen for all $f \in L^1(\mathbb{R}^n)$ (take for instance $x \mapsto e^{-\pi x^2}$ and $y \mapsto e^{-\pi(y-1)^2}$). The proof that $h$ is surjective can be easily derived from the result in Section 9.22 of W. Rudin - Real and Complex Analysis. Now let’s prove that $h$ is an homeomorphism. Call $\tau$ the topology of the one-point compactification of $\mathbb{R}^n$ and $\gamma$ the usual Gelfand topology on $\Delta$. To prove that
$(\mathbb{R}^n, \tau)$ is topologically equivalent to $(\Delta, \gamma)$ we just need to note that $h(\tau)$ is a compact Hausdorff topology and $\gamma \subset h(\tau)$. Since $h$ is one-to-one, $h(\tau)$ is a compact Hausdorff topology. To prove the inclusion remember that the Gelfand topology of $\Delta$ is just the weak topology induced by $\hat{\mathcal{L}}$, that is, the weakest topology that makes every $\Lambda (f + \beta \delta) \in \hat{\mathcal{L}}$ continuous. Since, for all $\Lambda (f + \beta \delta) \in \hat{\mathcal{L}}$

\[ \Lambda (f + \beta \delta) = [\alpha \mapsto \alpha (f + \beta \delta)] \]

\[ = [h_t \mapsto h_t (f + \beta \delta) = \hat{f} (t) + \beta], \]

then $h(\tau)$ makes every $\Lambda (f + \beta \delta) \in \hat{\mathcal{L}}$ continuous, hence, unsurprisingly, the weak topology $\gamma$ is weaker than $h(\tau)$. Since $\gamma \subset h(\tau)$ and both topologies are compact and $T_2$, it follows $\gamma = h(\tau)$.

**Remark 137.** Proposition on the preceding page proves that in $\mathcal{L}$, $\Delta = \overline{\mathbb{R}^n}$. Hence for all $f + \alpha \delta \in \mathcal{L}$ and for all $t \in \mathbb{R}$,

\[ \Lambda (f + \alpha \delta) (t) = \Lambda (f + \alpha \delta) (h_t) = \hat{f} (t) + \alpha. \]

This shows how the Fourier transform of an $L^1$ function coincides with its Gelfand transform\(^1\), making the Gelfand transform a proper generalization of the Fourier transform.

### 2.2 Lomonosov’s Invariant Subspace Theorem (Tommaso Russo)

In what follows $X$ is a Banach Space and we denote

\[ \mathcal{B}(X) = \{ T : X \to X \ \text{linear and bounded} \}. \]

Note that $X$ is not assumed to be a Banach Algebra, indeed the theory seen in the course will not be applied to $X$, but to $\mathcal{B}(X)$, which has a clear structure of Banach Algebra with unit. What is missing in this Algebra is commutativity, however some of the results that we discussed still hold. In particular we mention that the following result holds also with no assumptions of commutativity and will have a very main role in the sequel.

**Theorem 138 (Spectral Radius Formula).** Let $A$ be a Banach Algebra with unit and let $x \in A$. Then the *spectral radius* of $x$, defined by

\[ \rho(x) := \max_{\lambda \in \sigma(x)} |\lambda| \]

satisfies the following *Spectral Radius Formula*

\[ \rho(x) = \lim_{n \to +\infty} \|x^n\|^{1/n}. \]

\(^1\) If $\alpha = 0$, the equation above says that $\Lambda f = \mathcal{F} f$. 
Definition 139. Let $T \in \mathcal{B}(X)$. $M \subseteq X$ is said to be an invariant subspace for $T$, if $M$ is a closed subspace of $X$ and $T(M) \subseteq M$. It's said to be nontrivial if $M \neq \{0\}, M \neq X$.

The main topic of this section will be to discuss the existence of nontrivial invariant subspaces for elements of $\mathcal{B}(X)$. These are quite interesting in Operator Theory according to the following trivial remark: if $M$ is an invariant subspace for $T$, then $T|_M \in \mathcal{B}(M)$ and $M$ is a Banach Space too, so one can study the operator $T$ on a smaller space and then try to put together the behaviour of the operator on the various invariant subspaces in order to deduce the behaviour on the whole space. This is a very familiar machinery since is exactly what one does in finite dimensional spaces when tries to diagonalize a matrix or to find its Jordan Canonical Form. However in infinite dimensional spaces, it may happen that an operator has no eigenvalues (even if $X$ is a Complex Banach Space) and so one can not use eigenspaces, while generic invariant subspaces could be present. As a simple (but useful for what follows) example consider:

Example 140. Let $X = \ell^2$ and $S_R \in \mathcal{B}(\ell^2)$ the right shift, defined by:

$$(S_R x)_k = \begin{cases} 0 & k = 1 \\ x^{k-1} & k \geq 2 \end{cases} \quad \forall x \in \ell^2$$

if $x \in \ell^2$ is the sequence $x = (x^k)_{k=1}^{+\infty}$. Explicitly the action is the following:

$$(x^1, x^2, x^3, \cdots) \mapsto (0, x^1, x^2, x^3, \cdots).$$

If one tries to solve $S_R x = \lambda x$, it comes out $\lambda x^1 = 0$, $\lambda x^{k+1} = x^k, \forall k \geq 1$. Now there are two cases: $\lambda = 0$ immediately implies $0 = x^k, \forall k \geq 1$, while $\lambda \neq 0$ gives $x^1 = 0$ and then inductively $x^k = 0, \forall k$. Hence there are no eigenvalues.

For the sake of completeness, let us mention that $\sigma(S_R) = \overline{D(0,1)}$. Some evidence of this is given by the fact that $S_R^N$ has norm 1, $\forall N \geq 1$ and so the Spectral Radius Formula gives $\sigma(S_R) \subseteq \overline{D(0,1)}$.

Then it's natural to ask the following question: fix $X$ Banach Space and $T \in \mathcal{B}(X)$. Is it true that $T$ admits a non trivial invariant subspace? Under which conditions on $T$ and or on $X$ there exists an invariant subspace? Let's begin by some cases in which the answer is easy and we can answer with no condition on $T$, in particular the answer is the same for all elements of $\mathcal{B}(X)$. So in these cases we actually solve a more difficult problem\textsuperscript{2}: given $X$ Banach Space, is it true that every $T \in \mathcal{B}(X)$ admits an invariant subspace?

- In $\mathbb{R}^2$ the answer is no. Indeed it suffices to consider the operator given by the following matrix (wrt the canonical basis of course)

$$
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
$$

\textsuperscript{2}This is sometimes referred to as the Invariant Subspace Problem.
Every nontrivial invariant subspace would have dimension 1 and so would be an eigenspace, but there are no real eigenvalues and so no nontrivial invariant subspaces.

- Instead in $\mathbb{R}^n$ $n \geq 3$ the answer is yes; this is proved by standard (and quite boring) linear algebra arguments.

- In $\mathbb{C}^n$ the answer is trivial, since by the Fundamental Theorem of Algebra the characteristic polynomial has roots, so there are eigenspaces.

- The first infinite dimensional case that can be treated easily is when $X$ is non separable and the answer is yes also here. To see this, take $x_0 \in X$ different from zero and consider the closed linear span of $\{T^n x_0\}_{n=0}^{\infty}$ which is closed, invariant and separable, hence a proper subspace.

Now we have some example in which we ask something also on the operator:

- If we start with a Hilbert Space $\mathcal{H}$, then it follows form the Spectral Theorem that every normal operator admits nontrivial invariant subspaces.

- In 1954 Aronszajn and Smith proved that every compact operator on a complex Banach Space admits a non trivial invariant subspace.

- In 1966 Bernstein and Robinson gave a generalization of the above result: if there exists a non zero polynomial $p$ such that $p(T)$ is compact, then $T$ has a non trivial invariant subspace. Their proof used nonstandard analysis, a proof of the same result using only classical concepts was given by Halmos in the same year.

Now we turn to the key result, from which the last two statements follow immediately. The original proof, given by Lomonosov in 1973, made use of the Schauder Fixed Point Theorem in order to show the existence of an eigenvalue; the proof to be presented here instead deduces the same from the Spectral Radius Formula, this argument was proposed by Hilden in 1977.

**Theorem 141** (Lomonosov’s Invariant Subspace Theorem). Let $X$ be a infinite-dimensional complex Banach Space and let $T \in \mathcal{B}(X)$ be a nonzero compact operator. Then there exists a closed proper subspace $M \subseteq X$ which is invariant for all $S \in \mathcal{B}(X)$ that commute with $T$. In formula:

$$S \in \mathcal{B}(X), \quad ST = TS \Rightarrow S(M) \subseteq M$$

Before proving the statement, let us make some comment:

- In particular we have that every operator which commutes with a nonzero compact one admits a proper invariant subspace.

---

\(^3\)Do not get surprised that non separability makes things easier: if the space is bigger, there are more subspaces, so is easier some to be invariant.

\(^4\)i.e. $T \in \mathcal{B}(\mathcal{H})$ such that $TT^* = T^*T$, where $T^*$ is the adjoint of $T$. 
2.2 Lomonosov’s Invariant Subspace Theorem (Tommaso Russo) 44

- Of course we can choose \( S \) to be equal to \( T \) and so we deduce that every compact operator admits a nontrivial invariant subspace, so we have Aronszajn and Smith result.

- Also the other statement we mentioned is easy to deduce: indeed we have two cases. If \( p(T) \neq 0 \), then it is a nontrivial compact operator which of course commutes with \( T \) and so by Lomonosov Theorem \( T \) admits a proper invariant subspace. If \( p(T) = 0 \), let \( p(z) = \sum_{n=0}^{N} \alpha_n z^n \) with \( \alpha_N \neq 0 \), \( N \geq 1 \) so that we have \( 0 = \sum_{n=0}^{N} \alpha_n T^n \), hence \( T^N = -\sum_{n=0}^{N-1} \frac{\alpha_n}{\alpha_N} T^n \).

By this it’s clear that the linear span of \( \{ T^n(x_0) : n = 0, \cdots , N - 1 \} \) for some \( 0 \neq x_0 \in X \) is invariant under \( T \) and is closed and proper since is finite dimensional.

**Proof.** Denote by \( \Gamma := \{ S \in \mathcal{B}(X) : ST = TS \} \), which is easily seen to be a subalgebra of \( \mathcal{B}(X) \) and for \( y \in X \) denote \( \Gamma_y := \{ S y : S \in \Gamma \} \). These are subspaces of \( X \) (since \( \Gamma \) is a subspace of \( \mathcal{B}(X) \)) and are invariant under all \( S \in \Gamma \) (since \( \Gamma \) is closed under composition). They may miss being closed, but \( S(\Gamma_y) \subseteq \Gamma_y \) implies\(^5\) \( S(\Gamma_y) \subseteq \Gamma_y \), so that \( \forall y \in X, \Gamma_y \) is closed, invariant under all \( S \in \Gamma \) and \( \Gamma_y \neq \{ 0 \} \) if \( y \neq 0 \). If for at least one \( y \neq 0, \Gamma_y \subseteq X \) the proof is concluded, otherwise we have that \( \forall y \in X, y \neq 0, \Gamma_y \) is dense in \( X \). Let’s assume this.

Now, we use that \( T \) is nonzero: there exists \( x_0 \neq 0 \) in \( X \) such that \( T(x_0) \neq 0 \), so \( \| T(x_0) \| > \frac{1}{2} \| Tx_0 \| > 0 \), \( \| x_0 \| > \frac{1}{2} \| x_0 \| > 0 \) and by continuity of \( T \) there exists \( B \) open ball centered at \( x_0 \) such that \( \forall x \in B \), we have \( \| Tx \| \geq \frac{1}{2} \| Tx_0 \| \), \( \| x \| \geq \frac{1}{2} \| x_0 \| \). In particular \( 0 \notin K := \overline{T(B)} \) which is compact, since \( T \) is a compact operator. By our previous assumption \( \forall y \in K, \Gamma_y \cap B \neq \emptyset \), so there exists \( S_y \in \Gamma \) such that \( S_y y \in B \) and by continuity of \( S_y \) there exists an open neighborhood \( W_y \) of \( y \) s.t. \( S_y(W_y) \subseteq B \). Of course \( \{ W_y \}_{y \in K} \) is an open cover of \( K \), so it admits a finite subcover: there exist open sets \( W_1, \cdots , W_n \) whose union covers \( K \) and \( S_1, \cdots , S_n \in \Gamma \) with \( S_i(W_i) \subseteq B \), \( \forall i = 1, \cdots , n \).

Start with \( x_0 \in B \), so that \( T x_0 \in K \), hence in some \( W_i \), and so \( x_1 := S_{i_1} T x_0 \in B \); then \( T x_1 \in K \), so in some \( W_{i_2} \) and \( x_2 := S_{i_2} T x_1 \in B \). Going on in this way, we find a sequence \( x_N = S_{i_N} \cdots S_{i_1} T x_0 \in B \), so that \( \forall N \| x_N \| \geq \frac{1}{2} \| x_0 \| \). But of course by commutativity \( x_N = S_{i_N} \cdots S_{i_1} T^N x_0 \) and this implies

\[
\frac{1}{2} \| x_0 \| \leq \| x_N \| \\
\leq \| S_{i_N} \| \| S_{i_{N-1}} \| \cdots \| S_{i_1} \| \| T^N \| \| x_0 \| \\
\leq \| T^N \| \| x_0 \| \mu^N
\]

if \( \mu := \max \{ \| S_1 \|, \cdots , \| S_n \| \} \). Thus we have

\[
\frac{1}{2^N} \leq \mu \| T^N \| \mu^N, \forall N \geq 1
\]

\(^5\) \( S \) is continuous, so \( S(\overline{A}) \subseteq \overline{S(A)} \)
and we can pass to limit and use the Spectral Radius Formula to deduce

\[ 1 \leq \mu \rho(T) \]

which gives

\[ \rho(T) \geq \frac{1}{\mu} > 0. \]

The proof is now almost concluded, it suffices to recall some spectral properties for compact operators. What we need is the following: \( \sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} \), i.e. every element of the spectrum different from 0 is an eigenvalue; moreover it has finite multiplicity. So here we deduce that \( T \) admits an eigenvalue \( \lambda \neq 0 \) and the corresponding eigenspace \( M_\lambda := \{ x \in X : Tx = \lambda x \} \) is finite dimensional hence is closed and proper. Finally \( M_\lambda \) is invariant under all \( S \in \Gamma \), since if \( S \in \Gamma \) and \( x \in M_\lambda \) we have \( TSx = STx = S\lambda x = \lambda Sx \), so \( Sx \in M_\lambda \). This concludes the proof. \( \Box \)

As we have previously seen, this is quite a striking result. However it doesn’t settle the invariant subspace problem: indeed there exist operators which commute with no nontrivial compact operator, for example the right shift already introduced.

**Example 142.** Let as before \( S_R \in \mathcal{B}(\ell_2) \) and let \( T \in \mathcal{B}(\ell_2) \) be compact, such that \( S_R T = TS_R \). Then \( T = 0 \).

Indeed let \( N > M \geq 1 \) and compute, for \( x \in X \),

\[ \| TS_R^M x - TS_R^N x \| = \| S_R^M Tx - S_R^N Tx \|. \]

Set \( y := Tx \) and note\(^6\)

\[ S_R^M y - S_R^N y = (0, 0, \ldots, 0, y_1^1, y_2^2, y_3^3, \ldots) - (0, 0, \ldots, 0, y_1^1, y_2^2, y_3^3, \ldots) = (0, 0, \ldots, 0, y_1^1, y_2^2, y_3^3, \ldots, y_{N-M}^{N-M}, y_{N-M+1}^{N-M+1}, \ldots, y_{N-M+2}^{N-M+2}, \ldots). \]

Hence,

\[ \| TS_R^M x - S_R^N Tx \|^2 = \sum_{j=1}^{N-M} |y_j|^2 + \sum_{j=1}^{+\infty} |y^{N-M+j} - y_j|^2 \]

\[ \geq \sum_{j=1}^{N-M} |y_j|^2 \]

and now we can let \( N-M \to +\infty \) and deduce

\[ \liminf_{N-M \to +\infty} \| TS_R^M x - TS_R^N x \|^2 \geq \sum_{j=1}^{+\infty} |y_j|^2 = \| Tx \|^2. \]

\(^6\)probably the notation here is not the most beautiful, but is hopefully clear.
But \( \{S^n R x\}_{n \in \mathbb{N}} \) is a bounded sequence and so \( \{TS^n R x\}_{n \in \mathbb{N}} \) is precompact and hence \( TS^n R x \to \tilde{x} \) as \( k \to +\infty \). Passing to another subsequence if necessary we can assume \( n_{k+1} - n_k \to +\infty \) as \( k \to +\infty \), so that \( N = n_{k+1}, M = n_k \) in the previous inequality gives

\[
0 = \|\tilde{x} - \tilde{x}\| = \liminf_{k \to +\infty} \|TS^{n_k} R x - TS^{n_{k+1}} R x\|^2 \geq \|Tx\|^2.
\]

Thus we have \( Tx = 0, \forall x \in X \) so \( T = 0 \), which is what we wanted to prove.

Note that, although we can not use Lomonosov Theorem to deduce the existence of invariant subspaces, it’s clear that \( S_R \) admits many invariant subspaces, for example \( M_k := \{ x = (x^i)_{i=1}^{\infty} \in \ell_2 : x^1 = \cdots = x^k = 0 \} \). This is an interesting case also because all invariant subspaces of \( S_R \) are known, this is a theorem by Beurling.

In fact operators with no nontrivial invariant subspaces have been found: for example Enflo in 1987 built a non reflexive space and an operator on it without nontrivial invariant subspaces and Read in 1989 produced examples in the Classical Spaces \( c_0 \) and \( \ell_1 \). So in all cases the space \( X \) was non reflexive, under the assumptions of reflexivity, no counterexample has been found and the question is still open, in particular even in Hilbert Spaces.
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